

CfA Plasma Talks

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Notes from a series of 13 one hour (or more) lectures on Plasma Physics given to Ramesh Narayan' research group at the Harvard-Smithsonian Center for Astrophysics, between January and July 2012.

Lectures 1 to 5 cover various key Plasma Physics themes. Lectures 6 to 12 mainly go over the Review Paper on “Multidimensional electron beam-plasma instabilities in the relativistic regime” [*Physics of Plasmas* **17**, 120501 (2010)]. Lecture 13 talks about the so-called Biermann battery and its ability to generate magnetic fields from scratch.

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1 Introduction

When is a gas ionized?

- Ionization can come from the plasma itself, if hot enough. With $X = Ne/N_{\text{neutral}}$, Saha equation gives,

$$\frac{X^2}{1-X} = \frac{1}{nh^2} (2\pi m_e k_B T)^{3/2} e^{-I/k_B T}, \quad (1)$$

where I is the Ionization energy. Comes from statistical physics inside atom + Maxwell distribution outside. $X \rightarrow 0$ for $k_B T \lll I$, and $X \rightarrow 1$ for $k_B T \ggg I$.

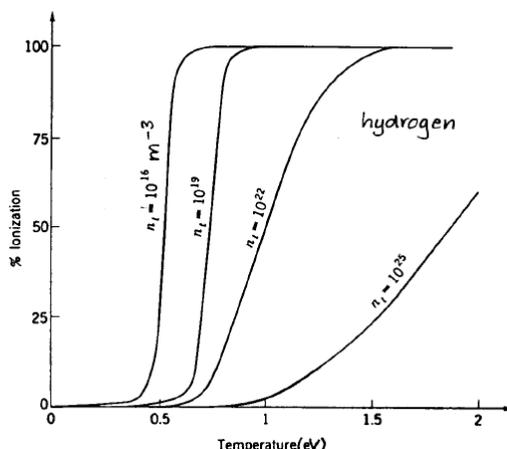


Figure 1: Degree for Ionization for Hydrogen, with, $I = 13.6$ eV.

- Ionization can come from external medium (Ionosphere ? $T = \text{say } 1000$ K).
- Ionization can come from the proximity of atoms ? Share electrons : metal.

Classification

Say temperature T , density N .

Typical distance between two electrons: $N^{-1/3}$.

Typical Coulomb energy: $e^2/N^{-1/3}$.

Typical kinetic energy in classical regime: $k_B T$.

More kinetic energy than Coulomb: $k_B T > e^2/N^{-1/3}$. Big frontier.

Classical relativistic: $k_B T > mc^2$.

Then come *quantum* effects. When $T >$ Fermi temperature T_F , with $k_B T_F = \hbar^2(3\pi^2 N)^{2/3}/2m_e$.

Thus, for $T < T_F$, energy increases with *density*, not *temperature*.

$k_B T_F < mc^2$, k_B scales like $N^{2/3}$.

$k_B T_F > mc^2$, k_B scales like $N^{1/3}$ (White Dwarfs to Neutron Stars?).

Important quantities

Time it takes to neutralize charge in-balance: *Plasma frequency*

$$\omega_p^2 = \frac{4\pi N e^2}{m_e} = 9000 \sqrt{N} \text{ [cm}^{-3}\text{]}. \quad (2)$$

That's why some waves bounce against the ionosphere.

Distance over which charge in-balance can exist: *Debye length*

$$\lambda_D = \frac{V_{th}}{\omega_p} = \sqrt{\frac{k_B T}{4\pi N e}} = 7.43 \cdot 10^2 \sqrt{T \text{ [K]}/N \text{ [cm}^{-3}\text{]}} \text{ [cm]}. \quad (3)$$

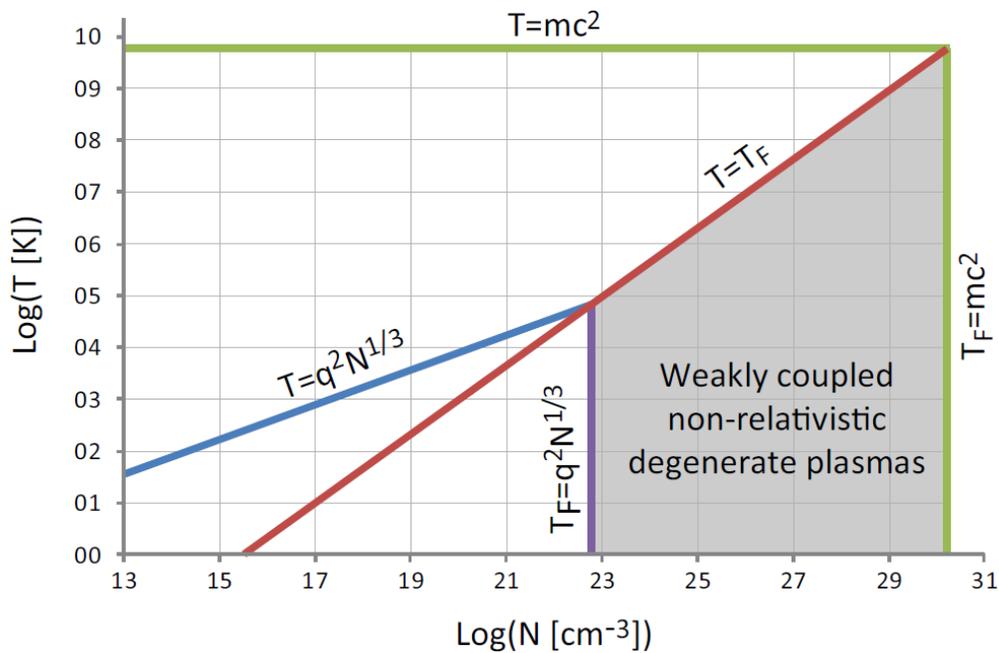


Figure 2: Classification of plasmas.

2 Kinetic theory

Vlasov and Boltzmann equations

Say only electrons + fixed positive background. Most basic description level. $G(\mathbf{r}, \mathbf{p}, t)d^3\mathbf{r}d^3\mathbf{p}$ = number of particles in $d^3\mathbf{r}d^3\mathbf{p}$ around (\mathbf{r}, \mathbf{p}) at time t . How does it evolve?

Say a particle has momentum \mathbf{p} at position \mathbf{r} , at time t . Say a force \mathbf{F} acts on it. At time $t + dt$, it will have momentum $\mathbf{p} + \mathbf{F}dt$, and position $\mathbf{r} + \mathbf{p}/\gamma m dt$ (*non-quantum* treatment). Therefore, ALL particles in $d^3\mathbf{r}d^3\mathbf{p}$ around (\mathbf{r}, \mathbf{p}) at time t MUST be in $d^3\mathbf{r}d^3\mathbf{p}$ around $(\mathbf{r} + \mathbf{p}/\gamma m dt, \mathbf{p} + \mathbf{F}dt)$ at time $t + dt$. That means,

$$G\left(\mathbf{r} + \frac{\mathbf{p}}{\gamma m}dt, \mathbf{p} + \mathbf{F}dt, t + dt\right) = G(\mathbf{r}, \mathbf{p}, t) \quad (1)$$

The “hyper” volume element $d^3\mathbf{r}d^3\mathbf{p}$ does not change (Jacobian = 1 here). Just Taylor expand the left-hand-side to get the **Vlasov Equation**,

$$\begin{aligned} G(\mathbf{r}, \mathbf{p}, t) + \frac{\partial G}{\partial \mathbf{r}} \cdot \frac{\mathbf{p}}{\gamma m}dt + \frac{\partial G}{\partial \mathbf{p}} \cdot \mathbf{F}dt + \frac{\partial G}{\partial t}dt &= G(\mathbf{r}, \mathbf{p}, t), \\ \Rightarrow \frac{\partial G}{\partial \mathbf{r}} \cdot \frac{\mathbf{p}}{\gamma m} + \frac{\partial G}{\partial \mathbf{p}} \cdot \mathbf{F} + \frac{\partial G}{\partial t} &= 0. \end{aligned} \quad (2)$$

Now, this result is NOT always right. Why?

We have assumed the force \mathbf{F} does not change over dt . But \mathbf{F} is an *averaged* force, in the same way the function G is averaged (IGM = 10^{-6} part/cm⁻³. If not averaged, $d^3\mathbf{r}$ must be gigantic, not infinitesimal). What if something “un-smooth” happened during dt ?

Close collisions are local, quasi-instantaneous processes, sending some particles OUT of $d^3\mathbf{p}$ around \mathbf{p} , and some other particles INSIDE $d^3\mathbf{p}$ around \mathbf{p} , during dt . (Think about billiard ball collisions: local and instantaneous). We’ll then have,

$$\begin{aligned} G(\mathbf{r}, \mathbf{p}, t) + \frac{\partial G}{\partial \mathbf{r}} \cdot \frac{\mathbf{p}}{\gamma m}dt + \frac{\partial G}{\partial \mathbf{p}} \cdot \mathbf{F}dt + \frac{\partial G}{\partial t}dt \\ = G(\mathbf{r}, \mathbf{p}, t) + [\text{Collisions}(\mathbf{r}, t), \mathbf{q} \rightarrow \mathbf{p}, \forall \mathbf{q}] - [\text{Collisions}(\mathbf{r}, t), \mathbf{p} \rightarrow \mathbf{q}, \forall \mathbf{q}], \end{aligned} \quad (3)$$

giving the **Boltzmann Equation**,

$$\frac{\partial G}{\partial \mathbf{r}} \cdot \frac{\mathbf{p}}{\gamma m} + \frac{\partial G}{\partial \mathbf{p}} \cdot \mathbf{F} + \frac{\partial G}{\partial t} = \int_{\mathbf{q}} [\text{Collisions}(\mathbf{r}, t), \mathbf{q} \rightarrow \mathbf{p}] - [\text{Collisions}(\mathbf{r}, t), \mathbf{p} \rightarrow \mathbf{q}]. \quad (4)$$

The right-hand-side, referred to as the “collision term”, is analitically untractable. Yet, that’s the one driving the relaxation to a Maxwellian distribution $G_M \propto e^{-v^2}$. For practical purposes, alternative forms have been worked-out (Fokker-Planck/Landau, Balescu, Krook $\nu(G_M - G)\dots$).

Vlasov or Boltzmann ?

In a plasma, particles are influenced by,

- Close collisions, changing \mathbf{p} rapidly and appreciably (say $\theta > \pi/2$). Accounted for by the collision term in the kinetic equation.
- “Distant” collisions, which amount to the influence of the overall plasma ($\rho, \mathbf{J} \rightarrow \mathbf{E}, \mathbf{B}$). Accounted for by the Force term in the kinetic equation.

Define a “close” collision by closest approach¹ $< R_L$, such as $e^2/R_L = E_K$ where E_K is the typical Kinetic energy ($k_B T$ or $k_B T_F$). Frequency for such collisions is roughly $\nu \sim n R_L^2 v_K$, with $m v_K^2 = E_K$. Time scale for “distant” collisions if $\sim \omega_p^{-1}$. Vlasov’s equation, with collision term = 0, is valid for $\nu \ll \omega_p$, i.e.,

$$n \left(\frac{e^2}{E_K} \right)^2 \sqrt{\frac{E_K}{m}} \ll \sqrt{\frac{4\pi n e^2}{m}} \Leftrightarrow e^2 n^{1/3} \ll E_K, \quad (5)$$

which just defines *weakly coupled plasmas*, where there is more kinetic energy than Coulomb potential energy, whether degenerate or not.

The Vlasov-Maxwell system

For weakly coupled plasmas, the first equation needed is therefore Vlasov’s with $\mathbf{F} = \text{Lorentz}$,

$$\frac{\partial G}{\partial t} + \frac{\mathbf{p}}{\gamma m} \cdot \frac{\partial G}{\partial \mathbf{r}} + q \left[\mathbf{E}(\mathbf{r}, t) + \frac{\mathbf{v}}{c} \times \mathbf{B}(\mathbf{r}, t) \right] \cdot \frac{\partial G}{\partial \mathbf{p}} = 0. \quad (6)$$

System is closed with Maxwell’s equations, where charge and current densities are given by,

$$\begin{aligned} \rho(\mathbf{r}, t) &= \int G(\mathbf{r}, \mathbf{p}, t) d^3 \mathbf{p}, \\ \mathbf{J}(\mathbf{r}, t) &= \int q G(\mathbf{r}, \mathbf{p}, t) \mathbf{v} d^3 \mathbf{p}. \end{aligned} \quad (7)$$

Eqs. (6,7), together with Maxwell’s, form the *Vlasov-Maxwell* closed system of equations. In 1D along axis x , we just have for $G(x, p, t)$ and $E(x, t)$,

$$\frac{\partial G}{\partial t} + \frac{p}{\gamma m} \frac{\partial G}{\partial x} + q E \frac{\partial G}{\partial p} = 0, \quad \frac{\partial E}{\partial x} = 4\pi q \int G(x, p, t) dp, \quad (8)$$

with $q = -e$ for electrons. Landau damping comes from these 2, originally with $\gamma = 1$.

¹Subscript L stands (again) for Landau.

3 From Kinetic to Fluid to MHD Equations

From Kinetic to Fluid

Fluid equations can be deduced from the *moments* of the kinetic equation¹. The fluid macroscopic density n , velocity \mathbf{v} and pressure tensor \mathbf{P} are defined through,

$$\begin{aligned} n(\mathbf{r}, t) &= \int F(\mathbf{r}, \mathbf{u}, t) d^3\mathbf{u}, & \mathbf{P}(\mathbf{r}, t) &= \int m(\mathbf{u} - \mathbf{v}) \otimes (\mathbf{u} - \mathbf{v}) F(\mathbf{r}, \mathbf{u}, t) d^3\mathbf{u}, \\ n(\mathbf{r}, t)\mathbf{v}(\mathbf{r}, t) &= \int \mathbf{u} F(\mathbf{r}, \mathbf{u}, t) d^3\mathbf{u}, \end{aligned} \quad (1)$$

where \otimes is “dyadic” product $\mathbf{u} \otimes \mathbf{v} = (u_i v_j)$. If our plasma is *cold*, which *kinetically* means $F(\mathbf{r}, \mathbf{u}, t) = \delta[\mathbf{u} - \mathbf{v}(\mathbf{r}, t)]G(\mathbf{r}, t)$, the density $n(\mathbf{r}, t)$ and the velocity $\mathbf{v}(\mathbf{r}, t)$ are what we would expect. Interestingly enough, the pressure tensor vanishes. *Microscopic velocity spread translates to macroscopic pressure*. Consider now the non-relativistic Vlasov kinetic equation,

$$\frac{\partial F}{\partial t} + \mathbf{v} \cdot \frac{\partial F}{\partial \mathbf{r}} + \frac{\mathbf{E} + \mathbf{v} \times \mathbf{B}/c}{m} \cdot \frac{\partial F}{\partial \mathbf{v}} = 0. \quad (2)$$

The moments of the equation give,²

$$\begin{aligned} \int [\text{Vlasov}] d^3\mathbf{p} &\Rightarrow \frac{\partial n}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (n\mathbf{v}) = 0, \\ \int m\mathbf{u} [\text{Vlasov}] d^3\mathbf{p} &\Rightarrow mn \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{r}} \right) = qn \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) - \frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{P}. \end{aligned} \quad (3)$$

For isotropic pressure³ with $\mathbf{P} = p\mathbf{I}$, the last term is just the usual gradient $\partial p/\partial \mathbf{r} = \nabla p$. The “convective derivative” term $(\partial_t + \mathbf{v} \cdot \nabla)$ simply *follows* a fluid element.

At this stage, you can close the system introducing a relation between $n(\mathbf{r}, t)$ and $p(\mathbf{r}, t)$, that is, an *equation of state*. Like for the first moment and the pressure, the Vlasov moment $\int u^n(\) d^3u$ always yields a macroscopic quantity $\propto \int u^{n+1}(\) d^3u$ from the $\mathbf{v} \cdot \partial F/\partial \mathbf{r}$ term. Still regarding the micro/macro duality: a *non-zero collision term* in the Vlasov equation is needed to recover *viscosity* or *friction* on the macro level.

From Fluid to MHD

We have initially one distribution function $F_i(\mathbf{r}, \mathbf{u}, t)$ per species. The procedure above shows we eventually have one set of fluid equations per species. Assume we just have protons and

¹See the Appendix of Spitzer’s *Physics of Fully Ionized Gases* for details. Also, Chapter I of William L. Kruer, *The Physics of Laser Plasma Interactions* (Previewed on *Google Books*).

²Not straightforward. See Kruer for details. Note that $\partial/\partial \mathbf{r}$ is an alternative notation for ∇ .

³If the pressure tensor is anisotropic, with $\mathbf{P} = (p_{i,j})$,

$$\frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{P} = \left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z}, \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z}, \frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z} \right).$$

electrons of densities $n_p(\mathbf{r}, t)$ and $n_e(\mathbf{r}, t)$. If we want to describe fast phenomenon where electrons could be decoupled from protons (faster than ω_p^{-1} , or smaller than λ_D), we need to keep two sets of equations. The so-called *Braginskii Equations* might be the most elaborate version of this option.

What if we're interested in slow $\tau \ll \omega_p^{-1}$, and large scale $\gg \lambda_D$, effects? Electrons are expected to closely follow protons. The plasma is a electron/proton "soup". Electroneutrality on these scales gives $n_p(\mathbf{r}, t) \sim n_e(\mathbf{r}, t)$. In the same way we defined the fluid quantities (1) and found they obey Eqs. (3), we define the MHD variables,

$$\begin{aligned}\rho(\mathbf{r}, t) &= m_p n_p(\mathbf{r}, t) + m_e n_e(\mathbf{r}, t), & \mathbf{V}(\mathbf{r}, t) &= \frac{m_e \mathbf{v}_e + m_p \mathbf{v}_p}{m_e + m_p} \\ \mathbf{J}(\mathbf{r}, t) &= q n_p(\mathbf{r}, t) \mathbf{v}_p(\mathbf{r}, t) - q n_e(\mathbf{r}, t) \mathbf{v}_e(\mathbf{r}, t).\end{aligned}\quad (4)$$

Combining the fluid equations for electrons and protons yields⁴,

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho \mathbf{V})}{\partial \mathbf{r}} = 0, \quad (5)$$

$$\rho \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \frac{\partial \mathbf{V}}{\partial \mathbf{r}} \right) = \frac{\mathbf{J}}{c} \times \mathbf{B} - \nabla \left(\overbrace{p_i + p_e}^P \right) + \rho \mathbf{g}, \quad (6)$$

where $\rho \mathbf{E}$ is neglected with respect to the Lorentz force, as $n_e \sim n_p \Rightarrow \mathbf{E} \sim 0$. Also, a gravity term $\rho \mathbf{g}$ is added here. Its fluid counterpart in Eq. (3) would obviously be $n m \mathbf{g}$. The system is closed through,

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}, \quad \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathcal{E}}{\partial t}. \quad (7)$$

Inserting $\mathbf{J} = c \nabla \times \mathbf{B} / 4\pi$ into Eq. (6) gives the usual magnetic pressure and tension terms. The last equation used to close the system is Ohm's law, which simplest version reads

$$\mathbf{J} = \sigma \left(\mathbf{E} + \frac{\mathbf{V}}{c} \times \mathbf{B} \right), \quad (8)$$

where σ is the medium conductivity. This equation is just $\mathbf{J} = \sigma \mathbf{E}$ in the fluid-frame at velocity \mathbf{V} , transformed in the Lab. frame⁵. *Ideal* MHD sets $\sigma = \infty$, so that $\mathbf{E} = -\mathbf{V} \times \mathbf{B} / c$. Two concluding remarks:

- Yes, we sometime consider $\mathbf{E} = 0$, like in Eqs. (6) and (7-right), and sometime $\mathbf{E} \neq 0$ like in Ohm's law or (7-left). Kulsrud⁶ explains well how this proves reasonable.
- We've cheated a little bit. We use the collisionless Vlasov's equation, and then talk about EOS or Ohm's law, which *imply collisions*. It's just far simpler to forget about collisions at the *kinetic/micro* level, derive the fluid equations, and then get collisions back into the game, kind of empirically, at the *fluid/macro* level.

⁴Eqs. (3) formally give a non-linear term $n_p m_p (\mathbf{v}_p \cdot \nabla) \mathbf{v}_p + n_e m_e (\mathbf{v}_e \cdot \nabla) \mathbf{v}_e \neq \rho (\mathbf{V} \cdot \nabla) \mathbf{V}$. An "=" is obtained neglecting the electron momentum, and considering $\mathbf{V} \sim \mathbf{v}_p$.

⁵J.D. Jackson, *Classical Electrodynamics*, p. 472.

⁶R.M. Kulsrud, *Plasma Physics for Astrophysics*, p. 44.

4 Linear Landau damping - The Maths

Just a piece of a vast problem: Energy exchange between waves and particles in a plasma. Simply put, in terms of the energy transfer direction:

- Waves \rightarrow Particles: Particle acceleration, wave damping.
- Particles \rightarrow Waves: Wave instability.

The original paper is Ref. [1]. Landau damping is one of the most studied/debated problem in plasma physics. Nice Maths *and* Physical derivation¹.

Calculation overview

Since the calculation is quite subtle and long, it may be useful to get a general overview from the very beginning. Here are the steps we will follow:

1. Derivation of the dispersion equation $\epsilon(k, \omega) = 0$ from the 1D Vlasov-Poisson system.
2. Landau contour, the continuity requirement and the Laplace transform.
3. Resolution for small damping and any distribution function.
4. Maxwellian distribution.

Dispersion Equation

Start from 1D non-relativistic equations² for $F(x, v = p/m, t)$ and field $E(x, t)$,

$$0 = \frac{\partial F}{\partial t} + v \frac{\partial F}{\partial x} - e \frac{E}{m} \frac{\partial F}{\partial v}, \quad (1)$$

$$\frac{\partial E}{\partial x} = 4\pi e \left[n_0 - \int F(x, v, t) dv \right], \quad (2)$$

where $n_0 = \int F_0 dv$ is the equilibrium density. Assume $F = F_0 + F_1$, with $|F_1| \ll |F_0|$, F_0 being an equilibrium solution. Same for E . The equilibrium electric field $E_0 = 0$. Linearizing Eqs. (1,2), assuming $F_1, E_1 \propto \exp(ikx - i\omega t)$, gives

$$0 = -i\omega F_1 + ikvF_1 - e \frac{E_1}{m} \frac{\partial F_0}{\partial v}, \quad (3)$$

$$ikE_1 = -4\pi e \int F_1(x, v, t) dv. \quad (4)$$

¹See Kip Thorne's Caltech course "Applications of Classical Physics", Chapter 21 mostly for the Maths part at <http://www.pma.caltech.edu/Courses/ph136/yr2004/>.

²Easily generalized to 3D.

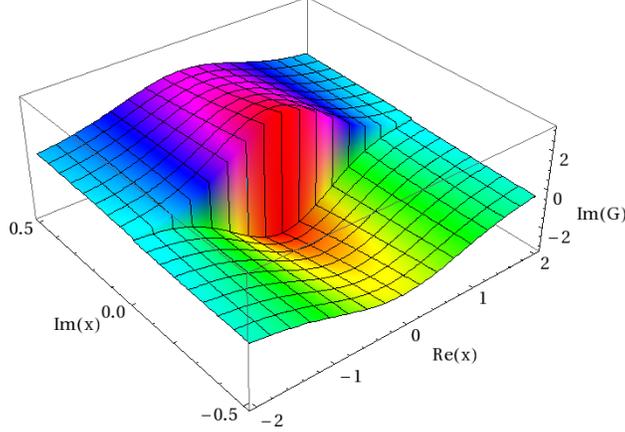


Figure 3: Imaginary part of $G = \int e^{-u^2} du/u - x$, for $x \in \mathbb{C}$. The real axis is a discontinuity.

Extract F_1 from the first equation, and plug it into the second,

$$\begin{aligned} \epsilon(k, \omega) &= 0, \quad \text{with,} \\ \epsilon(k, \omega) &= 1 - \frac{\omega_p^2}{k^2} \int \frac{f'_0}{v - \omega/k} dv, \end{aligned} \quad (5)$$

where $\omega_p^2 = 4\pi n_0 e^2/m$, $f_0 = F_0/n_0$ and $f'_0 = \partial f_0/\partial v$. This dispersion relation was first obtained by Vlasov in 1925 [2]. It shows ω should be imaginary. Otherwise, we have a problem, unless $f'_0(\omega/k) = 0$. The dielectric function $\epsilon(k, \omega)$ has therefore a real *and* an imaginary part, which for all kind of systems, is related to *dissipation*.

We could just consider ω imaginary and take this quadrature as it is, integrating along the real axis. But there's a problem. The resulting function of ω is *discontinuous*, precisely when crossing the real axis. As an illustration, Fig. 3 displays the imaginary part of $G = \int e^{-u^2} du/u - x$, for $x \in \mathbb{C}$. The discontinuity is obvious around $\text{Im}(x) = 0$. One part of the plan has to be physically meaningful, and the other not. But which one? We could try both options, and check that damping comes only when choosing the upper one. But what if we didn't know in the first place that a Maxwellian is stable? We shall see that a Laplace analysis of the problem can fully answer the question, and will indeed tell us that the "physical" half-plane is the upper one.

Admitting for now the upper-plane is the physical one, what do we do with the lower one? The answer is that we have to "analytically continue" the function we have on the upper-plane, to the lower one. This means finding a function on the lower plane which makes a continuous, "analytical" junction, with what we have on the upper one. In this respect, a uniqueness theorem from complex analysis helps: if somehow we find an expression in the lower plane matching what we have in the upper one, then this is the only one. The Landau contour is going to do all of that for us: providing a contour of integration equivalent to an integration over the real axis for $\text{Im}(\omega) > 0$, and an analytical continuation of the later in the lower plane $\text{Im}(\omega) < 0$.

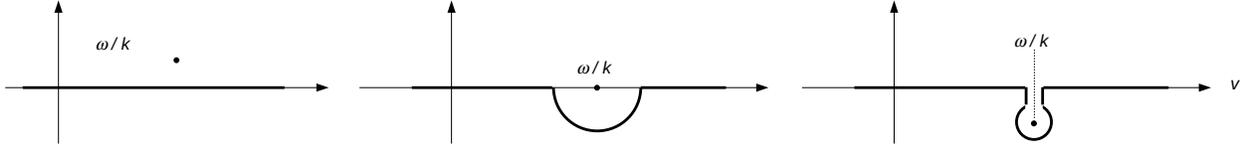


Figure 4: The Landau integration contour. It is not closed. It lies always on the same side of the pole. It lies below the pole.

Landau contour, the continuity requirement and the Laplace transform

Let's first give the solution found by Landau, namely the famous “Landau contour”. Figure 4 shows this integration contour has 3 very distinctive features:

1. The Landau contour is *not* closed by the “usual” semi-circle in the lower or upper half-plane.
2. The pole $\omega/k = (\omega_r + i\delta)/k$ must always lie on the *same side* of the Landau contour.
3. So, which side? The Landau contour goes *below* the pole.

These contour prescriptions are called the “Landau prescriptions”, and the corresponding contour, the “Landau contour”. We thus rewrite from now on Eq. (5) as

$$\epsilon(k, \omega) = 1 - \frac{\omega_p^2}{k^2} \int_L \frac{f'_0}{v - \omega/k} dv, \quad (6)$$

where \int_L means integration along the Landau contour. Let's now find out about these 3 features.

The contour is not closed

The contour just goes from $v = -\infty$ to $+\infty$, and is *not* closed in the upper or lower half-plane, as “usual”, because we have no guarantee $f'_0(v)$ behaves correctly there, so as to cancel the integration on the semi-circle at infinite radius. Indeed, considering a Maxwellian with $f'_0 \propto e^{-v^2}$ and setting $v = R_v e^{i\theta_v}$ to parameterize the integration on a circle of radius R_v , we find $f'_0 \propto e^{-R_v^2 \cos 2\theta_v}$ which can hardly be considered a vanishing quantity at $R_v \rightarrow \infty$ for *any* $\theta_v \in [0, \pi]$ (or $[0, -\pi]$, if you close in the lower half-plane).

Always on the same side

Assume $\omega = \omega_r + i\delta$, with $\delta > 0$. As long as δ remains positive in Eq. (5), the calculation does not pose any conceptual problem as the pole is not on the real axis, and continuity is guaranteed.

Now, what if δ approaches 0, and the pole ω/k even gets to cross the real axis? We would like $\epsilon(k, \omega)$ to be a *continuous* function of ω . Assume first we leave the integration contour unchanged (the real axis for v), and compare the quadrature for $\omega = \omega_r + i\delta$ and $\omega = \omega_r - i\delta$. The influence of the pole is mostly felt where the denominator is minimum at $v \sim \omega_r/k$, so let's locally get f'_0 out of the integral and compare,

$$I_1 = \int \frac{dv}{v - (\omega_r + i\delta)/k} \quad \text{and} \quad I_2 = \int \frac{dv}{v - (\omega_r - i\delta)/k}. \quad (7)$$

The difference $I_1 - I_2$ is,

$$I_1 - I_2 = 2i \int \frac{\delta/k}{(v - \omega_r/k)^2 + (\delta/k)^2} dv. \quad (8)$$

The continuity of $\epsilon(k, \omega)$ demands the expression above vanishes when $\delta \rightarrow 0^+$. The problem is that it does *not*. Instead, the quadrature tends to π (see function G_2 in Appendix A), so that we indeed have a *jump* of amplitude $2i\pi$ when crossing the real axis³.

The only way to avoid this is to deform the integration contour in such a way that it always lies *on the same side* of the pole ω/k .

The contour goes below the pole

To understand why the contour goes *below* the pole and not above like in Fig. 6, we need to follow Landau in rethinking the problem in terms of the time evolution of a perturbation applied at $t = 0$. The Fourier technique is not well suited for that because it entails an integration from $t = -\infty$ to $+\infty$. By design, it does not single out any special moment in between. By contrast, the *Laplace transform* involves times only from *zero* to $+\infty$. As shall be checked, the Laplace transform technique gives an unambiguous response about the location of the pole with respect to the integration contour.

Considering a function $h(t)$, its Laplace transform $\widehat{h}(\omega)$ and the inversion formula⁴, read

$$\widehat{h}(\omega) = \int_0^\infty e^{i\omega t} h(t) dt, \quad (10)$$

$$h(t) = \int_{C_L} e^{-i\omega t} \widehat{h}(\omega) d\omega, \quad (11)$$

where the contour C_L pictured on Fig. 5, passes above *all* the poles of $\widehat{h}(\omega)$ at height $\sigma > 0$, and can be closed in the lower half-plane where $e^{-i\omega t}$ behaves conveniently as to cancel the

³It can also be said that for I_1 , the integration path makes a *counter*-clockwise half-turn around the pole, so that $I_1 = i\pi$. But for I_2 , the half-turn around the pole is clockwise, so that $I_2 = -i\pi$ and $I_1 - I_2 = 2i\pi$.

⁴I here follow Landau's book, [3], p. 139, in defining the Laplace transform this way. That's just the usual one,

$$g(p) = \int_0^\infty g(t) e^{-pt} dt, \quad (9)$$

for $p = -i\omega$. It avoids having to rotate everything in the complex plane to relate the calculation to Eq. (5).

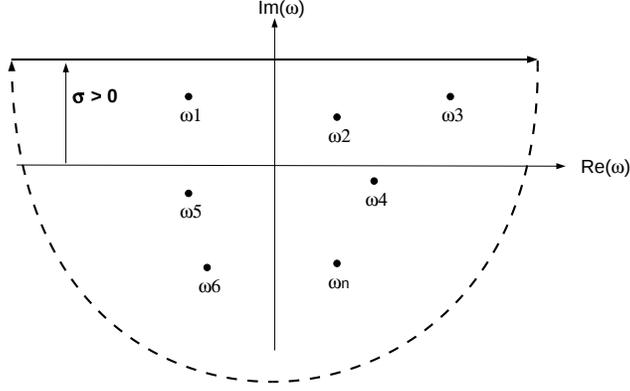


Figure 5: Laplace integration contour. Goes from $\omega = -\infty + i\sigma$ to $+\infty + i\sigma$ with $\sigma > 0$, and is closed in the lower half-plane. By design, $\sigma > 0$ and such that every single poles $\omega_1, \dots, \omega_n$ of the integrand lie *inside* the contour.

integral at infinity there. Note that although the requirement $\sigma > 0$ is emphasized in the book (p. 139), I still have to understand why being *above* all the poles is not enough. And as we shall see very soon, $\sigma > 0$ is the key to the choice of the right part of the ω complex plane.

Let's compute from the Maxwell-Vlasov Eqs. (1,2) the time evolution of the system considering,

$$F(x, v, t) = n_0 f_0(v) + F_1(v, t) e^{ikx}, \quad (12)$$

$$E(x, t) = E_1(t) e^{ikx}, \quad (13)$$

assuming F_1, E_1 are first order quantities, and $F_1(v, t = 0) e^{ikx}$ is the perturbation initially applied. The linearized Vlasov equation reads,

$$\frac{\partial F_1(v, t)}{\partial t} + ikv F_1(v, t) - \frac{en_0}{m} E_1(t) f_0'(v) = 0. \quad (14)$$

If we multiply by $e^{i\omega t}$ and take the integral from $t = 0$ to $+\infty$, an integration by part on the time derivative term gives,

$$\begin{aligned} \int_0^\infty e^{i\omega t} \frac{\partial F_1(v, t)}{\partial t} dt &= [e^{i\omega t} F_1(v, t)]_0^\infty - i\omega \int_0^\infty e^{i\omega t} F_1(v, t) dt \\ &= -F_1(v, 0) - i\omega \hat{F}_1(v, \omega), \end{aligned} \quad (15)$$

where $\lim_{t \rightarrow \infty} e^{i\omega t} F_1(v, t) = 0$ has been assumed. On the one hand, the very existence of the Laplace transform of $F_1(v, \omega) = \int F_1(v, t) e^{i\omega t} dt$ implies it. On the other hand, a important conclusion of the paper is that for large times, $F_1(v, t) \propto e^{ikvt}$ (see [1] p. 452, and *Plasma Talk* 5). This point is discussed neither in the book, nor in the original paper. Using Eqs.

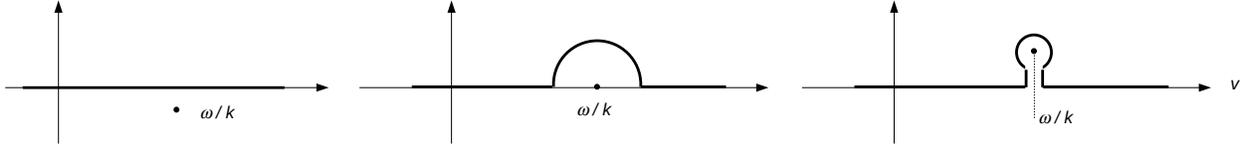


Figure 6: Forbidden option for the contour. Continuity is preserved, but the contour lies *above* the pole, in contradiction with the Laplace prescription.

(15,14) then gives,

$$(ikv - i\omega)\widehat{F}_1(v, \omega) - \frac{en_0}{m}\widehat{E}_1(\omega)f'_0(v) = F_1(v, 0), \quad (16)$$

where $F_1(v, 0)$ now acts like a “source term” at the right-hand-side. A few more manipulations exploiting Poisson’s equation (2) give,

$$\widehat{E}_1(\omega) = \frac{1}{\epsilon(k, \omega)} \frac{4\pi e}{k^2} \int_{-\infty}^{\infty} \frac{F_1(v, 0)dv}{v - \omega/k}, \quad (17)$$

where $\epsilon(k, \omega)$ is *identical* to Eq. (5). The time dependant electric field given by the inversion formula (11) is,

$$E_1(t) = \int_{C_L} e^{-i\omega t} \widehat{E}_1(\omega) d\omega = \int_{C_L} \frac{e^{-i\omega t}}{\epsilon(k, \omega)} \left[\frac{4\pi e}{k^2} \int_{-\infty}^{\infty} \frac{F_1(v, 0)dv}{v - \omega/k} \right] d\omega. \quad (18)$$

In contradistinction with Eq. (5) where the contour issue is puzzling, the Laplace technique used here is clear: The v -integration in $\epsilon(k, \omega)$ does go along the real axis, and the ω -integration is performed at fixed $\text{Im}(\omega) = \sigma > 0$. It means that in Eq. (18), which computes a physical quantity, the dielectric function $\epsilon(k, \omega)$ is calculated with ω *above* the real v -axis.

That answers the question we had: the physically meaningful half-plane we were wondering about after Eq. (5) is the *upper* one. The kind of contour pictured on Fig. 6 is thus “forbidden”.

Incidentally, what are the poles of the integrand in Eq. (18)? For “normal”, smooth initial excitations $F_1(v, 0)$, the term between brackets won’t have poles, so that our poles $\omega_1, \dots, \omega_n$ are *eventually* the zeros of $\epsilon(k, \omega)$. The ω -integration of Eq. (18) on the *closed* contour C_L will thus give, with $\omega_j = \omega_{r,j} + i\delta_j$,

$$E_1(t) = 2i\pi \sum_{j=1}^n \text{Res}(j) \equiv \sum_{j=1}^n A_j \exp(-i\omega_j t) = \sum_{j=1}^n A_j \exp(-i\omega_{r,j} t) e^{\delta_j t}, \quad (19)$$

which for large times will be governed by the largest δ_j . Therefore, the Laplace transform approach cannot spare us the resolution of $\epsilon(k, \omega) = 0$, as these zeros are the building blocks of the temporal response of the system.

Resolution for small damping

We suppose small damping, that is $|\delta| \ll |\omega_r|$, and Taylor expand Eq. (6),

$$\begin{aligned}
\epsilon(k, \omega_r + i\delta) &= \epsilon_r(k, \omega_r) + i\delta \left. \frac{\partial \epsilon_r}{\partial \omega_r} \right|_{\delta=0} + i \left[\epsilon_i(k, \omega_r) + i\delta \left. \frac{\partial \epsilon_i}{\partial \omega_r} \right|_{\delta=0} \right] \\
&= \epsilon_r(k, \omega_r) + i\epsilon_i(k, \omega_r) + \delta \left[i \frac{\partial \epsilon_r}{\partial \omega_r} - \frac{\partial \epsilon_i}{\partial \omega_r} \right]_{\delta=0} \\
&= \epsilon(k, \omega_r) + i\delta \left. \frac{\partial \epsilon_r}{\partial \omega_r} \right|_{\delta=0} + o(\delta),
\end{aligned} \tag{20}$$

where the $o(\delta)$ (negligible with respect to δ), comes from the fact that $\epsilon_i(\delta = 0) = 0$ (no damping, no dissipation, no imaginary dielectric function).

The first term $\epsilon(k, \omega_r)$ is given by Eq. (6) setting $\omega = \omega_r$, or taking the limit of $\epsilon(k, \omega_r + i\delta)$ for $\delta \rightarrow 0^+$. The part of the integration along the real axis for $v \in [-\infty, \omega_r/k - \varepsilon] \cup [\omega_r/k + \varepsilon, +\infty]$ gives the so-called ‘‘Cauchy Principal Part’’, denoted \mathbf{P} here. The part corresponding to the semi-circle (see Fig. 4 middle) gives the semi-residue for $v = \omega_r/k$. An alternative way of deriving this result, considering the limit $\delta \rightarrow 0^+$, is reported in Appendix A. We thus get,

$$\epsilon(k, \omega_r) = 1 - \frac{\omega_p^2}{k^2} \left[\mathbf{P} \int \frac{f'_0}{v - \omega_r/k} dv + i\pi f'_0(\omega_r/k) \right]. \tag{21}$$

This result allows to compute $\partial \epsilon_r / \partial \omega_r$ in Eq. (20), which eventually gives,

$$\epsilon(k, \omega) = 1 - \frac{\omega_p^2}{k^2} \mathbf{P} \int \frac{f'_0}{v - \omega_r/k} dv - i \frac{\omega_p^2}{k^2} \left[\pi f'_0(\omega_r/k) + \delta \frac{\partial}{\partial \omega_r} \mathbf{P} \int \frac{f'_0}{v - \omega_r/k} dv \right]. \tag{22}$$

Equating the real part to zero yields,

$$\frac{\omega_p^2}{k^2} \mathbf{P} \int \frac{f'_0}{v - \omega_r/k} dv = 1, \tag{23}$$

which was the result obtained by Vlasov in the first place. Canceling the imaginary part gives directly the damping rate,

$$\delta = -\pi \frac{f'_0(\omega_r/k)}{\frac{\partial}{\partial \omega_r} \mathbf{P} \int \frac{f'_0}{v - \omega_r/k} dv}. \tag{24}$$

Eqs. (23, 24) formally solve the problem in terms of the distribution function. A first order evaluation of $\mathbf{P} \sim k^2 / \omega_r^2$ (see Eq. (27) below), gives

$$\begin{aligned}
\omega_r &= \omega_p, \quad \text{and then} \\
\frac{\delta}{\omega_p} &= \frac{\pi \omega_p^2}{2 k^2} f'_0(\omega_p/k).
\end{aligned} \tag{25}$$

The rate δ has the sign of $f'_0(\omega_p/k)$. That means that if f_0 decreases for $v = \omega_p/k$, the waves is damped because $\delta < 0$. But if f_0 increases for $v = \omega_p/k$, we have $\delta > 0$ and the wave can actually *grow*.

One could argue we started initially assuming δ positive, and find it can be negative here. It is not a problem for the following reason: Eq. (22) we found assuming $\delta > 0$ is *continuous* at $\delta = 0$. It must therefore be identical to the integration *on the Landau contour* on both sides of the real axis. We can therefore confidently solve it regardless of the sign of δ . In other words, thanks to the Landau contour, we can compute the result as if δ was positive, and then don't care about the sign.

Historically, Vlasov first ran into Eq. (5). He escaped the problem posed by the pole on the real axis by considering only the \mathbf{P} of the quadrature. He did so apparently without much foundation, which Landau denounced without mercy in [1]. We understand from the analysis above that doing so, he missed the imaginary part which would have led to the ‘‘Vlasov damping’’.

Maxwellian distribution

Let's finally consider a 1D Maxwellian distribution,

$$f_0(v) = \frac{1}{\sqrt{2\pi k_B T/m}} e^{-mv^2/2k_B T}. \quad (26)$$

For phase velocities ω_r/k much larger than the thermal velocity $V_{th} = \sqrt{k_B T/m}$, we can expand the denominator in powers of kv/ω_r , since that quantity is small where the numerator is relevant. We thus have,

$$\begin{aligned} \mathbf{P} \int \frac{f'_0}{v - \omega_r/k} dv &= -\frac{k}{\omega_r} \int f'_0 \left(1 + \frac{kv}{\omega_r} + \frac{k^2 v^2}{\omega_r^2} + \frac{k^3 v^3}{\omega_r^3} + \dots \right) dv \\ &= \frac{k^2}{\omega_r^2} + 3 \frac{k_B T}{m} \frac{k^4}{\omega_r^4} + \dots \end{aligned} \quad (27)$$

For small k , namely $kV_{th}/\omega_r \sim kV_{th}/\omega_p \ll 1$, Eq. (23) now gives,

$$\omega_r^2 = \omega_p^2 (1 + 3k^2 \lambda_D^2), \quad \text{with} \quad \lambda_D = \frac{\sqrt{k_B T/m}}{\omega_p}. \quad (28)$$

We finally (phew!) use Eq. (24) to extract the damping rate. On the one hand, we compute the derivative of the \mathbf{P} with respect to ω_r using Eq. (27), and then simply set $\omega_r = \omega_p$ in the result. On the other hand, we set $\omega_r = \omega_p$ in f'_0 to find⁵

$$\delta = -\omega_p \frac{\sqrt{\pi/8}}{k^3 \lambda_D^3} \exp\left(-\frac{1}{2k^2 \lambda_D^2}\right). \quad (29)$$

Fluid theory just gives the real part of the frequency, namely Eq. (28), so that Landau damping is a purely kinetic effect.

⁵Some authors insert the full expression of ω_r from Eq. (28), yielding $-1/2k^2 \lambda_D^2 - 3/2$ in the argument of the exponential.

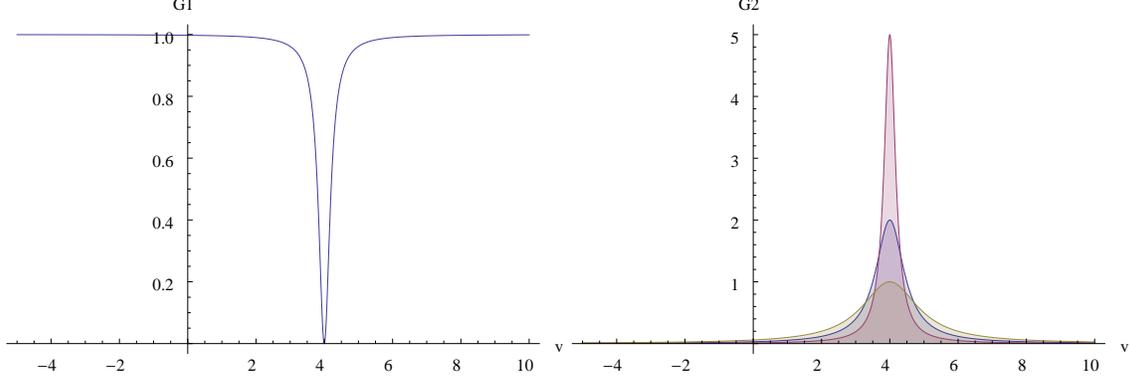


Figure 7: Functions G_1 and G_1 involved in Eq. (32). For small δ/k , G_1 is almost 1 everywhere, except for $v = \omega_r/k$, where it is 0. G_2 peaks at $v = \omega_r/k$ and tends to 0 elsewhere, while its integral is always π , like a Dirac δ function. Parameters are $\omega_r/k = 4$, $\delta/k = 0.2$ for G_1 , and $\delta/k = 0.2, 0.5, 1$ for G_2 .

Appendix A

Let's derive,

$$\int_L \frac{f'_0}{v - \omega_r/k} dv = \mathbf{P} \int \frac{f'_0}{v - \omega_r/k} dv + i\pi f'_0(\omega_r/k), \quad (30)$$

used for Eq. (21), without using the residue theorem. For $\omega = \omega_r + i\delta$ with $\delta > 0$, integration along the Landau contour is equivalent to an integration along the real axis. Let's thus assume $\delta > 0$ and compute,

$$I = \lim_{\delta \rightarrow 0^+} \int_{-\infty}^{\infty} \frac{f'_0}{v - (\omega_r + i\delta)/k} dv. \quad (31)$$

We multiply the numerator and the denominator of the integrand by $(v - \omega_r/k) + i\delta/k$, which is the complex conjugate of the denominator. We get an expression with a purely real, non singular denominator, and clearly separated real and imaginary parts,

$$I = \lim_{\delta \rightarrow 0^+} \int_{-\infty}^{\infty} \underbrace{\frac{(v - \omega_r/k)^2}{(v - \omega_r/k)^2 + (\delta/k)^2}}_{G_1} \frac{f'_0}{(v - \omega_r/k)} dv + i \int_{-\infty}^{\infty} \underbrace{\frac{\delta/k}{(v - \omega_r/k)^2 + (\delta/k)^2}}_{G_2} f'_0 dv. \quad (32)$$

Regarding the *real part*, the factor G_1 of the integrand is 0 for $v = \omega_r/k$, and ~ 1 for $\delta/k \ll |v - \omega_r/k|$. It tends to the $\mathbf{P} \int f'_0/(v - \omega_r/k)$ for small δ/k (see Fig. 7). The factor G_2 of the integrand of the *imaginary part* departs from 0 only for $v \sim \omega_r/k$. But its integral is always π . For small δ/k , the quadrature thus tends to $\pi f'_0(\omega_r/k)$, and we are back to (30)⁶.

⁶ The limit of $i\delta$ with $\delta \rightarrow 0^+$ is sometimes written “ $i0$ ”. The identity

$$\lim_{\delta \rightarrow 0^+} \int_{-\infty}^{\infty} \frac{h(x)}{x - a - i\delta} dx \equiv \int_{-\infty}^{\infty} \frac{h(x)}{x - a - i0} dx = \mathbf{P} \int_{-\infty}^{\infty} \frac{h(x)}{x - a} dx + i\pi h(a),$$

This calculation is consistent with the Landau contour integration *only* for $\delta \rightarrow 0^+$. This is because in such case, the real axis along which we perform the integration (32) coincide with the Landau contour. If we were to compute Eq. (32) for $\delta \rightarrow 0^-$, we would find the opposite imaginary part, just because in this case, the real axis *no longer* fits the Landau contour. The latter, instead, is deformed and keeps passing *below* the pole, precisely to avoid the discontinuity.

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can be referred to as the “Plemelj Formula” in the literature. For $\delta \rightarrow 0^-$, the imaginary part above is $-i\pi h(a)$.

5 Landau damping - The Physics, Plasma Echo, and a (little) word about the non-linear problem

While the original paper [1] is purely mathematical, a clearer physical picture is provided in Landau's book ([2], §30 p. 126). Suppose we switch on at $t = 0$ a 1D electrostatic wave $\mathbf{E} = E_0 \sin(kx - \omega t)\mathbf{x}$, traveling at $v_\phi = \omega/k$ along with a particle with velocity v_0 at $t = 0$. For v_0 slightly *larger* than v_ϕ , the particle is trapped in the wave potential, where it is going to oscillate. Doing so, it ends up with an average velocity close to the wave velocity v_ϕ . It should thus lose energy, and the energy goes to the wave. Situation is reversed for particles initially slightly *slower* than the wave. They end up gaining energy from the wave.

If slower particles are more numerous than the faster ones, the wave loses more than it gains, which means it is damped. Let's now "Fermi-calculate" this, not following Fermi but Jackson [3] and Spitzer [4] (who follows Jackson). Landau uses a slightly different approach, still implying a calculation with some small parameter eventually tending to zero. I chose Jackson¹ precisely because there's no such trick in his strategy. The reasoning is *non-relativistic*.

To start with, which particles can enter the game? If their velocity is too high *relatively* to the wave, they will flow from one potential crest to another, without much net energy exchange. The ones for which energy exchange is possible, are the ones which will be trapped by the potential. The wave potential height is,

$$\Delta\varphi = \frac{qE_0}{k}. \quad (1)$$

The maximum particle velocity Δv in the wave-frame at v_ϕ must then satisfy,

$$\frac{1}{2}m(\Delta v)^2 = \Delta\varphi \Rightarrow (\Delta v)^2 = \frac{2qE_0}{mk}. \quad (2)$$

Thus, only particles with velocity $v \in [v_\phi - \Delta v, v_\phi + \Delta v]$ in the lab-frame can be caught by the wave. This is pictured on Fig. 8, taken from another great work on Landau damping by Dawson [5]. For a particle near the center of this interval, we take $\sin x \sim x$ for the field, and the equation of motion in the wave-frame reads

$$m \frac{d^2x}{dt^2} + qE_0 kx = 0 \Rightarrow \frac{d^2x}{dt^2} + \omega_{osc}^2 x = 0, \quad \text{where } \omega_{osc}^2 = \frac{qE_0 k}{m}, \quad (3)$$

showing it oscillates in the wave potential with frequency ω_{osc} .

Before it was trapped, the particle energy in the lab-frame was just $W_1 = mv^2/2$. After the trapping, the energy is

$$W_2 = \frac{1}{2}m(v_\phi^2 + (v - v_\phi)^2), \quad (4)$$

where $\frac{1}{2}mv_\phi^2$ is the end translational kinetic energy, and $\frac{1}{2}m(v - v_\phi)^2$ can be viewed as an internal energy of oscillation². If there are $n_0 f_0(v)dv$ particles with such velocities, with

¹The J.D. Jackson who wrote *Classical Electrodynamics*.

²The energy of a mass oscillating in a potential is split between its kinetic energy and its potential energy. At the bottom of the potential well, all the energy is kinetic. The term $\propto (v - v_\phi)^2$ in Eq. (4) is the kinetic energy, in the wave frame, of the particle at the bottom of the well.

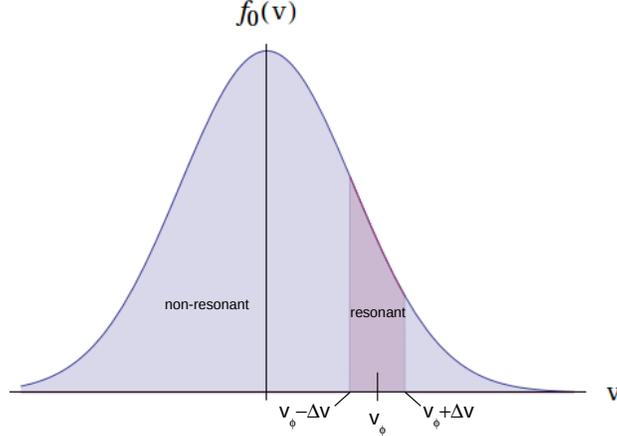


Figure 8: Division of the plasma between non-resonant and resonant (trapped) particles. Only resonant particles contribute to the calculation. After [5].

If $\int f_0 = 1$, the energy shift is

$$dW = n_0 f_0(v) dv (W_2 - W_1), \quad (5)$$

which we integrate over all particles capable of such exchange,

$$\Delta W = \int_{v_\phi - \Delta v}^{v_\phi + \Delta v} n_0 f_0(v) dv \frac{1}{2} m (v_\phi^2 + (v - v_\phi)^2 - v^2). \quad (6)$$

Expanding $f_0(v) = f_0(v_\phi) + (v - v_\phi) f'_0(v_\phi) + \dots$, the term corresponding to $f_0(v_\phi)$ in Eq. (6) vanishes³, and we find

$$\Delta W = -\frac{2}{3} m n_0 v_\phi \Delta v^3 f'_0(v_\phi). \quad (7)$$

If, like on Fig. 8, particles slower than v_ϕ are more numerous than faster ones, $f'_0(v_\phi) < 0$ and $\Delta W > 0$, which means particles *gain* energy at the expense of the wave. The wave is *damped*. We now just have to write that this energy leaves the field \mathbf{E} over a time scale ω_{osc}^{-1} ,

$$\frac{d(E_0^2/8\pi)}{dt} = -\omega_{osc} \Delta W = \omega_{osc} \frac{2}{3} m n_0 v_\phi \Delta v^3 f'_0(v_\phi). \quad (8)$$

Plugging here the expressions for Δv and ω_{osc} from Eqs. (2,3) we find,

$$\begin{aligned} \frac{d(E_0^2/8\pi)}{dt} &= \frac{8\sqrt{2}}{3} \omega \frac{\omega_p^2}{k^2} f'_0(v_\phi) \left(\frac{E_0^2}{8\pi} \right), \\ &\equiv 2\delta \left(\frac{E_0^2}{8\pi} \right). \end{aligned} \quad (9)$$

³It does *not* vanish if you omit the “internal energy” term in Eq. (4).

As the field energy $\propto E_0^2$ is damped at 2δ , the field itself is damped at δ . Setting finally $\omega = \omega_p$, we have

$$\frac{\delta}{\omega_p} = \frac{4\sqrt{2}\omega_p^2}{3k^2} f'_0(\omega_p/k), \quad (10)$$

identical to Eq. (25) of *Plasma Talk 4*, up to a numerical pre-factor close to 1 ($\pi/2 = 1.57$ and $4\sqrt{2}/3 = 1.88$). A discussion of the *non*-Galilean invariance of Eq. (10) is available in [3] (p. 180).

A word on Landau damping and gravitation

According to Ref. [6], Landau Damping of Gravitational Waves would *not* be possible. Much has been done with respect of Landau Damping of more mundane “gravity waves”. *The stability and vibrations of a gas of stars*, by Lynden-Bell, seems to be a quite influential paper [7]. The abstract concludes stating “Landau Damping occurs for wave-length smaller than the critical one [Jean’s]”.

Plasma Echo

Fascinating consequence of the fact that the *density* relaxes whereas the *distribution function* does *not* (many functions have the same integral). Original idea by Gould *et al.* [16].

Suppose we produce an initial electric field perturbation $\propto e^{-ik_1x}$ in the plasma. The Laplace analysis [1] of the *distribution function* temporal evolution (not the field, nor the density) shows it *indefinitely oscillates* with $F = f_0 + f_1(v) \exp(ik_1vt - ik_1x)$. For large times, any velocity integral “phase”-vanishes,

$$\lim_{t \rightarrow \infty} \int f_1(v) e^{ik_1vt - ik_1x} dv = 0, \quad (11)$$

which is how we recover zero field and density perturbations. The density perturbation and the field associated with f_1 die out, but f_1 *doesn't*. This is how you reconcile the necessary reversibility of the Vlasov-Maxwell system, with the apparent irreversibility of Landau Damping. There only seems to be a *macroscopic* irreversibility, but the evolution in *microscopically* reversible.

Is it possible to detect this ever oscillating $f_1(v)$ at later times ? Yes. Assume we wait for a time τ , and send another perturbation in the plasma $\propto e^{ik_2x}$. The second perturbation is going to modulate both f_0 and f_1 according to $e^{ik_2v(t-\tau) - ik_2x}$. Regarding f_1 , we will recover something varying like

$$e^{ik_1vt - ik_1x} e^{ik_2x - ik_2v(t-\tau)} = e^{i(k_2 - k_1)x + ik_2v\tau - i(k_2 - k_1)vt}. \quad (12)$$

The key-point here is that contrary to Eq. (11), where $k_1t \neq 0$ implies the velocity integral vanishes at large times, together with the first order density and field, the coefficient of v in the exponential above is exactly canceled at time,

$$t = \frac{k_2}{k_2 - k_1} \tau. \quad (13)$$

At this time, the velocity integral will *not* vanish, and an electric field should reappear in the plasma. So you perturb a plasma. You wait until everything apparently calmed down. Then you send another perturbation, and at the time prescribed by Eq. (13), an electric field will suddenly pop-up “out of nowhere”, related to the perturbation you first sent. That is the “plasma echo”.

The idea was experimentally tested soon after the theory came, and the echo was found [17]. Mouhot & Villani put it this ways: “A plasma which is apparently back to equilibrium after an initial disturbance, will react to a second disturbance in a way that shows that it has not forgotten the first one” ([14], p. 40).

Regarding gravitational systems, Lynden-Bell wrote “A system whose density has achieved a steady state will have information about its birth still stored in the peculiar velocities of its stars” ([7], p. 295).

***Nonlinear* Landau damping**

We found linear waves are damped. Landau Damping has been *experimentally* confirmed [8]. Here are a few landmarks for *large* amplitude waves (1D, non-relativistic)⁴:

- Isichenko 1997 [9]: Landau damping valid \forall amplitude (Theory).
- Manfredi 1997 [10]: Some large amplitude waves *do not* decay until $t = \infty$ (Numerical).
- Lancellotti & Dorning 1998 [11]: Existence of “critical initial states” for which $\lim_{t \rightarrow \infty} E \neq 0$ (Theory).
- Caglioti & Maffei 1998 [12]: Mathematical proof of the existence of *some* damped solutions (Theory).
- Medvedev *et al.* 1998 [13]: Damping of waves of finite amplitude and arbitrary shape according to $e^{\delta t}$, with $\lim_{t \rightarrow \infty} \delta = 0$ (Theory).

Mouhot & Villani 2010 [14, 15]: End of the controversy. Nonlinear Landau damping for general interactions, including Coulomb *and* Newton (therefore also including the case of galactic dynamics).

For any potential $V(\mathbf{r})$ such that $|V(\mathbf{k})| = O(|k|^{-2-\varepsilon})$, with $\varepsilon > 0$, and any *linearly* stable distribution function $f_0(x, v)$, large amplitude perturbations relax in such a way that all observables (density, field...),

$$\Psi(t) = \int f(t, x, v) \psi(x, v) dx dv, \tag{14}$$

relax exponentially with time. The distribution function itself does *not* relax to its value at $t = 0$. For small perturbations, $f(t, x, v)$ converges to something that is close to $f_0(x, v)$. For larger perturbations, the distribution function converges to something that is far from f_0 , or

⁴Thanks to Giovanni Manfredi for the summary!

it does not converge at all. The large time behavior of a strongly disturbed solution is still an open mystery.

See [14] for the full report, and a great history of the problem, or [15] for a shorter version. Villani was awarded the *2010 Fields Medal* for this.

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6 Beam Plasma Instabilities - Introduction

Miscellaneous

From now on, and for a number of Lectures, I'll just go through the Review Paper, "Multidimensional electron beam-plasma instabilities in the relativistic regime", *Physics of Plasmas*, **17**, 120501 (2010).

Counter-streaming flows, possibly relativistic. Lots of them. Basic system: counter-streaming *electron beams* with $n_{b0}, n_{p0}, \mathbf{v}_{b0}, \mathbf{v}_{p0}$ over a background of *fixed* protons n_i . Main hypothesis:

- Collisionless, Vlasov-Maxwell plasmas (i.e. weakly coupled, see *Plasma Talk 2*),
- Homogenous, no boundaries (system size $\gg c/\omega_p$),
- Initially current and charge neutral, $n_{b0}v_{b0} = n_{p0}v_{p0}$ and $n_{b0} + n_{p0} = n_i$,
- No \mathbf{B}_0 , to start with. . .

Motivations: simplest system + Fast Ignition Scenario for Inertial Fusion + Shock Acceleration physics (SNR's, GRB's). See Fig. 2 of Review Paper.

Particle-In-Cell Simulations: great tool for testing/guiding - See Fig. 3 of Review Paper.

A *multidimensional* unstable spectrum

- 1948: some perturbations with $\mathbf{k} \parallel$ to the flow are unstable - *Two-stream* modes.
- 1959: some perturbations with $\mathbf{k} \perp$ to the flow are unstable - *Filamentation* modes.
Still 1959: collisionless plasma with $T_x > T_y$, unstable for $\mathbf{k} \parallel \mathbf{y}$. *Weibel*.
Difference between them discussed in Sec. III. F of Review.
- 1960: some perturbations with \mathbf{k} arbitrarily oriented are unstable - *Oblique* modes.

Bottom line here: Which one will Nature choose? The fastest. Need to tackle the problem globally.

First: look at flow aligned, then flow-perp and the oblique modes. Second: which one grows faster?

7 Two-stream Instability

Two-stream (flow-aligned) modes

Interesting starting with a cold fluid 1D model. Equivalent to Vlasov with $f_0(v) \propto \delta(v - v_0)$. *Non-relativistic*. General case shows it's still relevant for the 3D case.

Linearize conservation and Euler equations. One set for each electron species, and I omit subscripts for clarity. Consider first orders quantities $n_{1p}, n_{1b}, E_1 \propto e^{ikx - i\omega t}$. Conservation and Euler equations read,

$$\frac{\partial n}{\partial t} + \frac{\partial(nv)}{\partial x} = 0, \quad (1)$$

$$m \frac{\partial v}{\partial t} + mv \frac{\partial v}{\partial x} = qE. \quad (2)$$

Once linearized, they respectively give

$$n_1 = n_0 \frac{kv_1}{\omega - kv_0}, \quad (3)$$

$$v_1 = i \frac{qE_1/m}{\omega - kv_0}, \quad (4)$$

so that,

$$n_1 = \frac{qn_0}{m} \frac{ikE_1}{(\omega - kv_0)^2}. \quad (5)$$

Then, from Poisson's equation¹

$$ikE_1 = 4\pi q(n_{1b} + n_{1p}), \quad (6)$$

we get,

$$1 = \frac{\omega_{pb}^2}{(\omega - kv_{0b})^2} + \frac{\omega_{pp}^2}{(\omega - kv_{0p})^2}, \quad \text{with } \omega_{p,bp}^2 = \frac{4\pi q^2 n_{0,bp}}{m}. \quad (7)$$

The frequency $\omega - kv$ is the Doppler shifted frequency. Can't help but thinking it looks like an energy conservation equation. Without drifts, $v_{0,bp} = 0$ and we just have

$$(\hbar\omega)^2 = (\hbar\omega_{pb})^2 + (\hbar\omega_{pp})^2. \quad (8)$$

Any ideas?

Until Eq. (5), species are disconnected from each other in the calculation. It is Poisson's equation which puts them together, summing the contribution of each species. Assume an infinite amount of these, each beamlet going at velocity v , with density $n_0 f_0(v) dv$, $\int f_0 = 1$. The extension of Eq. (7) reads,

$$1 = \int \frac{4\pi q^2 n_0 f_0(v) dv}{m(\omega - kv)^2} = \frac{\omega_{p0}^2}{k^2} \int \frac{f_0(v) dv}{(v - \omega/k)^2}, \quad \omega_{p0}^2 = \frac{4\pi q^2 n_0}{m}. \quad (9)$$

¹Poisson's equation brings a *vectorial* equation down to a *scalar* one. We thus loose information, unless $\mathbf{k} \cdot \mathbf{E} = kE$. The full 3D analysis shows modes with $k_{\perp} = 0$ are precisely like this.

identical with the one encountered in *Plasma Talk 4*, up to an integration by part.

So if we “Fermi understand” Eq. (7), we have everything.

Introducing the dimensionless variables,

$$x = \frac{\omega}{\omega_{pp}}, \quad Z = \frac{kv_{0b}}{\omega_{pp}}, \quad \alpha = \frac{n_{0b}}{n_{0p}}, \quad (10)$$

Eq. (7) reads,

$$1 = \frac{\alpha}{(x - Z)^2} + \frac{1}{(x + \alpha Z)^2}. \quad (11)$$

Diluted beam, $\alpha \ll 1$

There are techniques² to solve Eq. (11) in this regime, always approximately, for all Z . I just show here how to find *the* mode growing the most.

The beam is just a *perturbation* to the plasma. The modes of the system should be close to the modes of the plasma alone. We thus look for solutions at $\omega \sim \omega_{pp}$, i.e. $x = 1 + \epsilon$. We also know that the fastest growing mode should efficiently exchange energy with the beam. It should thus have $\omega/k \sim v_{0b}$. With $\omega \sim \omega_p$, that means $Z \sim 1$. Eq (11) now reads,

$$1 = \frac{\alpha}{(1 + \epsilon - Z)^2} + \frac{1}{(1 + \epsilon + \alpha Z)^2}. \quad (12)$$

As we’ll checked, $|1 - Z| \ll \epsilon$ and $\alpha Z \sim \alpha \ll \epsilon$, which gives

$$1 = \frac{\alpha}{\epsilon^2} + \frac{1}{(1 + \epsilon)^2} \quad \Rightarrow \quad 1 = \frac{\alpha}{\epsilon^2} + 1 - 2\epsilon \quad \Rightarrow \quad \epsilon^3 = \frac{\alpha}{2}. \quad (13)$$

By setting $\epsilon = \rho e^{i\theta}$, we find

$$\rho = \left(\frac{\alpha}{2}\right)^{1/3}, \quad \theta = -\frac{2\pi}{3}, 0, \frac{2\pi}{3}. \quad (14)$$

With $e^{\pm i2\pi/3} = -1/2 \pm i\sqrt{3}/2$, we obtain 3 modes

$$x = \frac{\omega}{\omega_{pp}} = 1 - \frac{\alpha^{1/3}}{2^{4/3}} - i\frac{\sqrt{3}}{2^{4/3}}\alpha^{1/3}, \quad (15)$$

$$= 1 - \frac{\alpha^{1/3}}{2^{4/3}}, \quad (16)$$

$$= 1 - \frac{\alpha^{1/3}}{2^{4/3}} + i\frac{\sqrt{3}}{2^{4/3}}\alpha^{1/3}, \quad \text{Unstable.} \quad (17)$$

As evidenced on Fig. 9, the most unstable mode has $Z \sim 1$, that is $k/\omega_{pp} \sim v_{0b}$. An electron from the beam always sees the same electric field.

How to compute these results in a Fermi-like way?

²See S. A. Bludman, K. M. Watson, and M. N. Rosenbluth, *Phys. Fluids* **3**, 747 (1960).

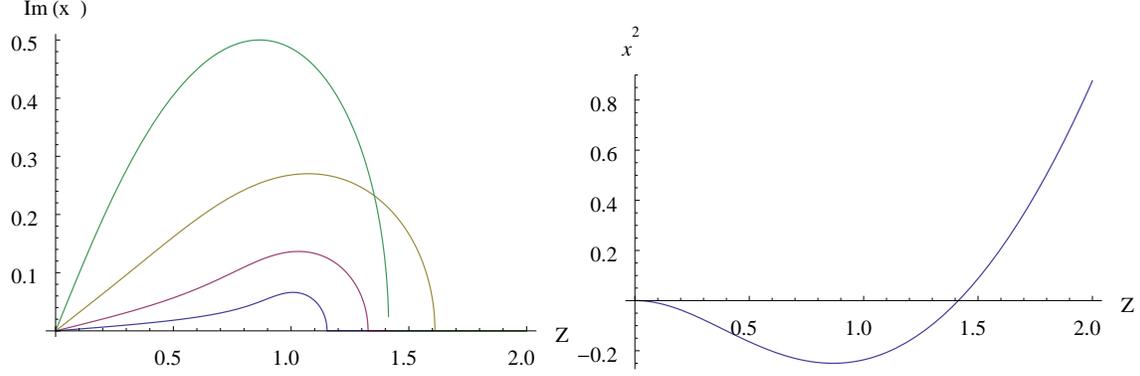


Figure 9: **Left:** Plot of $\text{Im}(x)$ in terms of Z for $\alpha = 10^{-3}, 10^{-2}, 10^{-1}$ and 1 . **Right:** Plot of Eq. (19), $x^2 = 1 + Z^2 - \sqrt{1 + 4Z^2}$. The system is unstable, $x^2 < 0$, for $Z < \sqrt{2}$.

Symmetric beams, $\alpha = 1$

Eq. (11) now reads,

$$1 = \frac{1}{(x - Z)^2} + \frac{1}{(x + Z)^2}, \quad (18)$$

which can be solved *exactly* for all Z , giving

$$x^2 = 1 + Z^2 \pm \sqrt{1 + 4Z^2}. \quad (19)$$

For $Z < \sqrt{2}$, the solution with a minus sign is unstable (see Fig. 9), with a most unstable wave-vector Z_m and its frequency x_m given by

$$Z_m = \frac{\sqrt{3}}{2}, \quad x_m = 0 + i\frac{1}{2}. \quad (20)$$

For the diluted beam regime, unstable modes are plasma Langmuir waves at $\omega \sim \omega_p$, traveling with the beam. Things are not so clear here. The beam is no longer a perturbation. The waves have $\text{Re}(\omega) = 0$, and are the modes of the full counter-streaming system “beam+plasma”, each of equal density.

To wrap-up the most unstable mode characteristics in terms of $\alpha \in [0, 1]$:

- Growth-rate: $\text{Im}(\omega/\omega_{pp}) = \frac{\sqrt{3}}{2^{4/3}}\alpha^{1/3} \rightarrow \frac{1}{2}$ (Note that $\frac{\sqrt{3}}{2^{4/3}} \sim 0.68 > \frac{1}{2}$).
- Frequency: $\text{Re}(\omega/\omega_{pp}) = 1 - \frac{\sqrt{3}}{2^{4/3}}\alpha^{1/3} \rightarrow 0$.
- Most unstable wave-vector: $Z = 1 \rightarrow \frac{\sqrt{3}}{2}$.

Relativistic effects

Maxwell’s and conservation equations are the same. Euler is now (subscripts omitted),

$$m \frac{\partial(\gamma v)}{\partial t} + mv \frac{\partial(\gamma v)}{\partial x} = qE. \quad (21)$$

It turns out that when linearizing “ γv ” instead of “ v ”, one finds,

$$\gamma v = \gamma_0 v_0 + v_1 \gamma_0^3 + \dots \quad (22)$$

As a result, Eq. (11) is replaced by,

$$1 = \frac{\alpha}{(x - Z)^2 \gamma_b^3} + \frac{1}{(x + \alpha Z)^2 \gamma_p^3}. \quad (23)$$

Intuitively, where does these $1/\gamma^3$ come from? If a particle oscillates along its main direction of motion, its mass gets a γ^3 relativistic boost. Changing m to $m\gamma^3$ in Eq. (7) gives the result above.

Diluted beam, $\alpha \ll 1$

Here, $\gamma_p \sim 1$, so that we can recycle the non-relativistic results for diluted beam, formally replacing $\alpha \rightarrow \alpha/\gamma_b^3$, i.e. $n_b \rightarrow n_b/\gamma_b^3$. The unstable modes given by Eq. (17) now reads,

$$x = \frac{\omega}{\omega_{pp}} = 1 - \frac{1}{2^{4/3}} \frac{\alpha^{1/3}}{\gamma_b} + i \frac{\sqrt{3}}{2^{4/3}} \frac{\alpha^{1/3}}{\gamma_b}. \quad (24)$$

Symmetric beams, $\alpha = 1$

With two symmetric beams, the Lorentz factors are the same $\gamma_p = \gamma_b \equiv \gamma$. Equation (23) now reads,

$$1 = \frac{1}{(x - Z)^2 \gamma^3} + \frac{1}{(x + Z)^2 \gamma^3}. \quad (25)$$

Here again, we just replace $x \rightarrow x\gamma^{3/2}$ and $Z \rightarrow Z\gamma^{3/2}$, and we're formally back to the non-relativistic case. Equation (20) then gives

$$Z_m = \frac{\sqrt{3}}{2\gamma^{3/2}}, \quad x_m = 0 + i \frac{1}{2\gamma^{3/2}}. \quad (26)$$

8 Filamentation Instability - Part 1

We still consider the same counter-streaming system, but look now at perturbations with $\mathbf{k} \perp$ to the flow. With respect to the Two-stream instability ($\mathbf{k} \parallel$ to the flow), the situation is reversed: The physics is simple, but the full maths are involved. Let's start with the physics.

Physical picture

Suppose two particle currents of same radius a and density n but opposite velocities u , perfectly overlap (Fig. 10, left). The system is charge and current neutral, in equilibrium. We now set them apart by a distance R (Fig. 10, right). The first current generates a \mathbf{B} field at the level on the second one. The field is such that the Lorentz force \mathbf{F} produced *repels* the other current even more. *Unstable* system. We can write,

$$F = dM \frac{d^2 R}{dt^2}, \quad (1)$$

where dM is the mass of the volume element. The force reads,

$$F = dq \frac{u}{c} B, \quad (2)$$

where dq is the charge of the volume element. With a density n , and particles of charge q and rest mass m , the charge dq and the mass dM of the volume dV read respectively,

$$dq = qndV, \quad \text{and} \quad dM = \gamma mndV, \quad (3)$$

where γm is the relativistic mass boost for transverse motion. Equation (1) now reads,

$$qndV \frac{u}{c} B = \gamma mndV \frac{d^2 R}{dt^2}, \quad \text{i.e.,} \quad q \frac{u}{c} B = \gamma m \frac{d^2 R}{dt^2}. \quad (4)$$

B is the field created by the current, so that

$$B = \frac{2I}{cR}, \quad \text{where} \quad I = nqu\pi a^2. \quad (5)$$

Replacing the current I by its expression, we find

$$\frac{d^2 \xi}{dt^2} = \frac{\delta^2}{\xi}, \quad \text{with} \quad \delta = \omega_p \frac{\beta}{\sqrt{2\gamma}}, \quad \text{and} \quad \xi = \frac{R}{a}, \quad \beta = \frac{u}{c}, \quad (6)$$

with $\omega_p^2 = 4\pi nq^2/m$. Although this equation won't give $\xi \propto e^{\delta t}$, it does tell the system does not relax to its initial state, on a time scale $\propto \delta^{-1}$, which fits *exactly* the result of the linear theory¹. Maybe an exponential grow would be obtained starting from opposite current *partially* overlapping.

¹Up to a factor of order unity, as usual.

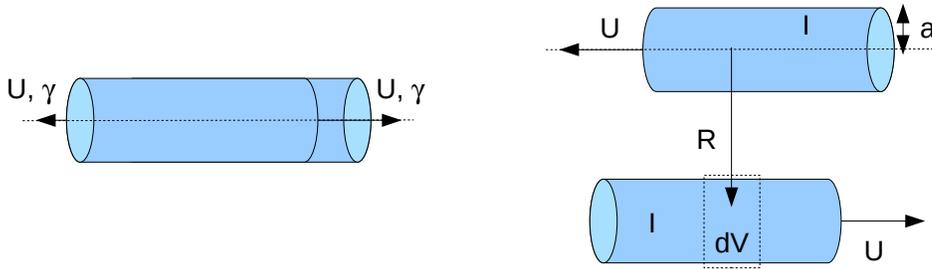
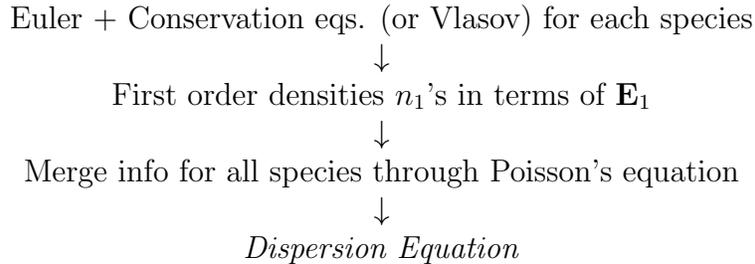


Figure 10: System unstable to the filamentation instability.

The Maths

The Dispersion Equation: Calculation Pattern

The dispersion equation for the filamentation instability is not easier to derive than the one for arbitrarily oriented \mathbf{k} 's. I will thus go over the general case $k_{\parallel}k_{\perp} \neq 0$, and then focus on $k_{\parallel} = 0$. For the Two-stream instability, the flow of the calculation was²:



We could use Poisson's equation for modes with $k \parallel$ to the flow because we know³ they have $\mathbf{k} \parallel \mathbf{E}$. In plasma jargon, we say these modes are *longitudinal*, or *electrostatic*. For the, Poisson equation, which convert a *vectorial* into a *scalar* identity, doesn't result in a loss of information, precisely because $\mathbf{k} \parallel \mathbf{E}$.

For filamentation modes, we don't know about the respective orientation of \mathbf{k} and \mathbf{E} . The divergence of the electric field would introduce the cosine of the $\widehat{\mathbf{k}, \mathbf{E}}$ angle, which is unknown. Poisson's $\nabla \cdot \mathbf{E} = 4\pi\rho$ gives $k_x E_x + k_y E_y + k_z E_z = 4\pi\rho$, which cannot be used as a dispersion equation because it yields only *one* equation for *three* components of the field. We need a 3D "merging species" equation which does not result in information loss. This is Maxwell-Ampère, which merges the *currents* instead of the *charges*.

The general pattern of the calculation is indeed quite similar:

²See *Plasma Talks 7*.

³We'll soon find out it is true.

Euler + Conservation eqs. (or Vlasov) for each species
 \downarrow
 First order *currents* \mathbf{J}_1 's in terms of \mathbf{E}_1
 \downarrow
 Merge info for all species through Maxwell-Ampère equations
 \downarrow
Dispersion Equation

Let's see this more in details, reasoning again from the fluid equations. Every equilibrium quantities are now slightly perturbed with terms $\propto \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t)$. The linearized conservation equations give for each species:

$$n_1 = n_0 \frac{\mathbf{k} \cdot \mathbf{v}_1}{\omega - \mathbf{k} \cdot \mathbf{v}_0}. \quad (7)$$

The linearized non-relativistic (so far) Euler equation give, still for each species:

$$\mathbf{v}_1 = \frac{-iq/m}{\omega - \mathbf{k} \cdot \mathbf{v}_0} \left(\mathbf{E}_1 + \frac{\mathbf{v}_0 \times \mathbf{B}_1}{c} \right). \quad (8)$$

It is easy to eliminate \mathbf{B}_1 through Maxwell-Faraday equation,

$$\mathbf{B}_1 = \frac{c}{\omega} \mathbf{k} \times \mathbf{E}_1, \quad (9)$$

so that we see how Eqs. (7,8) eventually give n_1 and \mathbf{v}_1 in terms of \mathbf{E}_1 alone, for each species,

$$\begin{aligned} \mathbf{v}_1 &= \frac{-iq/m}{\omega - \mathbf{k} \cdot \mathbf{v}_0} \left(\mathbf{E}_1 + \frac{\mathbf{v}_0 \times (\mathbf{k} \times \mathbf{E}_1)}{\omega} \right), \\ n_1 &= \frac{-iqn_0}{m} \frac{\mathbf{k}}{(\omega - \mathbf{k} \cdot \mathbf{v}_0)^2} \cdot \left(\mathbf{E}_1 + \frac{\mathbf{v}_0 \times (\mathbf{k} \times \mathbf{E}_1)}{\omega} \right). \end{aligned} \quad (10)$$

We may now write Maxwell-Ampère equation, to merge the information from all the species into one single equation depending of \mathbf{E}_1 only,

$$i\mathbf{k} \times \mathbf{B}_1 = \frac{-i\omega}{c} \mathbf{E}_1 + \frac{4\pi}{c} \mathbf{J}_1, \quad (11)$$

and eliminate \mathbf{B}_1 from Maxwell-Faraday Eq. (9) to obtain,

$$\frac{c^2}{\omega^2} \mathbf{k} \times (\mathbf{k} \times \mathbf{E}_1) + \mathbf{E}_1 + \frac{4i\pi}{\omega} \mathbf{J}_1 = 0. \quad (12)$$

The first order current is finally expressed through,

$$\mathbf{J}_1 = \underbrace{n_{0,b} \mathbf{v}_{1,b} + n_{1,b} \mathbf{v}_{0,b}}_{\text{Beam part}} + \underbrace{n_{0,p} \mathbf{v}_{1,p} + n_{1,p} \mathbf{v}_{0,p}}_{\text{Plasma part}}. \quad (13)$$

Although the end result is not really “user friendly”, we can see how Eqs. (10,12,13) eventually yield a *tensorial* equation of the form

$$\mathbf{T} \cdot \mathbf{E}_1 = 0. \quad (14)$$

When starting from the *Vlasov* equation, linearization gives the first order distribution function for each species,

$$f_1(\mathbf{k}, \mathbf{v}, \omega) = \frac{iq/m}{\omega - \mathbf{k} \cdot \mathbf{v}} \left(\mathbf{E}_1 + \frac{\mathbf{v} \times \mathbf{B}_1}{c} \right) \cdot \frac{\partial f_0}{\partial \mathbf{v}}. \quad (15)$$

Here again, Maxwell-Faraday Eq. (9) together with $n_1 = \int f_1 dv$ and $\mathbf{v}_1 = \int f_1 \mathbf{v} dv$, allow to reach the dispersion equation.

9 Filamentation Instability - Part 2

Dispersion Equation Analysis

The tensorial equation $\mathbf{T} \cdot \mathbf{E}_1$ at the end of *Plasma Talk 8* has the obvious solution $\mathbf{E}_1 = 0$. Now, the proper modes of our system are precisely the *non-trivial solutions* $\mathbf{T} \cdot \mathbf{E}_1 = 0$, with $\mathbf{E}_1 \neq 0$ ¹.

That tells us two things:

- If $(\exists \mathbf{E}_1 \neq 0 / \mathbf{T} \cdot \mathbf{E}_1 = 0) \Rightarrow \det \mathbf{T} = 0$. That's the *dispersion equation*, yielding ω in terms of \mathbf{k} .

Assume we pick up one wave vector \mathbf{k} . The dispersion equation

$$\det \mathbf{T}(\mathbf{k}, \omega) = 0, \tag{1}$$

gives one or more ω 's, $(\omega_{1,\mathbf{k}}, \dots, \omega_{N,\mathbf{k}}) \in \mathbb{C}^N$. Each couple $(\mathbf{k}, \omega_{j,\mathbf{k}})$ defines a proper mode of the system. Unstable modes have $\text{Im}(\omega) < 0$.

The fluid model usually gives a polynomial dispersion equation. Each new ingredient to the model (mobile ions, magnetic field, ...), adds waves. Polynomial of degree larger than 10 are common.

- The proper modes of the system $\mathbf{E}_1(\mathbf{k}, \omega)$ are in the *Kernel* of \mathbf{T} , which is precisely the set of non-zero \mathbf{E}_1 's fulfilling $\mathbf{T} \cdot \mathbf{E}_1 = 0$.

Assume again we picked up one wave vector \mathbf{k} . The dispersion equation gives a series of frequencies $(\omega_{1,\mathbf{k}}, \dots, \omega_{N,\mathbf{k}})$. We thus have N tensors with vanishing determinants. Each of these N tensors has a Kernel of dimension 1 or 2 (a Kernel of dimension 3 would imply $\mathbf{T}=\mathbf{0}$).

$$\begin{aligned} \mathbf{T}(\mathbf{k}, \omega_{1,\mathbf{k}}) &\Rightarrow \{\mathbf{E}_{1,i}(\mathbf{k}, \omega_{1,\mathbf{k}})\}_{i=1 \text{ or } 2}, \\ &\vdots \\ \mathbf{T}(\mathbf{k}, \omega_{N,\mathbf{k}}) &\Rightarrow \{\mathbf{E}_{1,i}(\mathbf{k}, \omega_{N,\mathbf{k}})\}_{i=1 \text{ or } 2}, \end{aligned}$$

So, for one couple $(\mathbf{k}, \omega_{\mathbf{k}})$, the formalism tells how is the \mathbf{E}_1 field. It lies either along a given direction, or in a plane. In particular, the formalism tells us about the $\widehat{\mathbf{k}, \mathbf{E}}$ angle. We don't have to assume waves are longitudinal² ($\mathbf{k} \parallel \mathbf{E}$), or transverse ($\mathbf{k} \perp \mathbf{E}$). The formalism *decides for us*.

For a flow $\parallel \mathbf{z}$, and $\mathbf{k} = (k_x, 0, k_z)$ as pictured on Fig. 11, the final form of the tensor \mathbf{T} is given by,

$$\mathbf{T} = \begin{vmatrix} \eta^2 \varepsilon_{xx} - k_z^2 & 0 & \eta^2 \varepsilon_{xz} + k_z k_x \\ 0 & \eta^2 \varepsilon_{yy} - k^2 & 0 \\ \eta^2 \varepsilon_{xz} + k_x k_z & 0 & \eta^2 \varepsilon_{zz} - k_x^2 \end{vmatrix}, \tag{2}$$

where $\eta = \omega/c$ and $\varepsilon_{\alpha\beta}$ is given by Eq. (8) of the Review Paper.

¹We could also say we look for the eigen-vectors associated with the eigen-value $\lambda = 0$.

²Also referred to as "electrostatic".

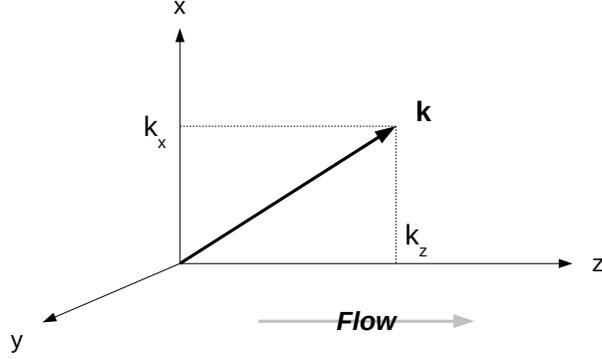


Figure 11: Axis conventions.

Two-stream Check

Let's check our assumption from *Plasma Talk 7*, that for $\mathbf{k} \parallel \text{flow}$, i.e. $k_x = 0$, there are longitudinal modes with $\mathbf{k} \parallel \mathbf{E}$. Setting $k_x = 0$ in Eq. (2) gives,

$$\mathbf{T}(k_z, k_x = 0) = \begin{vmatrix} \eta^2 \varepsilon_{xx} - k^2 & 0 & \eta^2 \varepsilon_{xz} \\ 0 & \eta^2 \varepsilon_{yy} - k^2 & 0 \\ \eta^2 \varepsilon_{xz} & 0 & \eta^2 \varepsilon_{zz} \end{vmatrix}. \quad (3)$$

For such wave vectors, the system is perfectly symmetric around the flow axis z . We thus have $\varepsilon_{xx} = \varepsilon_{yy} \equiv \varepsilon_{\perp}$, and³ $\varepsilon_{xz} = 0$, so that

$$\mathbf{T}(k_z, k_x = 0) = \begin{vmatrix} \eta^2 \varepsilon_{\perp} - k^2 & 0 & 0 \\ 0 & \eta^2 \varepsilon_{\perp} - k^2 & 0 \\ 0 & 0 & \eta^2 \varepsilon_{zz} \end{vmatrix}. \quad (4)$$

The equation $\mathbf{T} \cdot \mathbf{E}_1 = 0$ defines two kinds of waves:

- Assume (\mathbf{k}, ω) fulfills,

$$\varepsilon_{\perp} = k^2 c^2 / \omega^2, \quad (5)$$

then, ε_{zz} will in general *not* vanish for the *same* (\mathbf{k}, ω) . For these (\mathbf{k}, ω) , the tensor will thus have the form,

$$\mathbf{T} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \eta^2 \varepsilon_{zz} \neq 0 \end{vmatrix}, \quad (6)$$

and waves with with $\mathbf{E}_1 \in (x, y)$ satisfy $\mathbf{T} \cdot \mathbf{E}_1 = 0$. Since $\mathbf{k} = (0, 0, k_z)$, these are *transverse* modes, $\mathbf{k} \perp \mathbf{E}_1$. In general, they are stable.

³Less obvious, but true.

- If we consider now (\mathbf{k}, ω) fulfilling

$$\varepsilon_{zz} = 0, \quad (7)$$

we find non-zero solutions of $\mathbf{T} \cdot \mathbf{E}_1 = 0$ are waves with $\mathbf{E}_1 \in (z)$, as the tensor now takes the form,

$$\mathbf{T} = \begin{vmatrix} \eta^2 \varepsilon_{\perp} - k^2 \neq 0 & 0 & 0 \\ 0 & \eta^2 \varepsilon_{\perp} - k^2 \neq 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}. \quad (8)$$

Since $\mathbf{k} = (0, 0, k_z)$, these are *longitudinal* modes, $\mathbf{k} \parallel \mathbf{E}_1$, with dispersion equation, which indeed are our two-stream modes. It is thus checked that the modes we investigated in *Plasma Talk 7* do exist.

The Filamentation Instability

About the Dispersion Equation

Let's now consider $k_z = 0$ in Eq. (2). We find,

$$\mathbf{T} = \begin{vmatrix} \eta^2 \varepsilon_{xx} & 0 & \eta^2 \varepsilon_{xz} \\ 0 & \eta^2 \varepsilon_{yy} - k^2 & 0 \\ \eta^2 \varepsilon_{xz} & 0 & \eta^2 \varepsilon_{zz} - k^2 \end{vmatrix}, \quad (9)$$

where $\mathbf{T} \cdot \mathbf{E}_1 = 0$ again defines two kinds of modes:

- Modes with $\mathbf{E}_1 \in (y)$, therefore transverse since $\mathbf{k} \parallel \mathbf{x}$, with dispersion equation,

$$\varepsilon_{yy} = k^2 c^2 / \omega^2. \quad (10)$$

- The *Filamentation modes* (at last), with $\mathbf{E}_1 \in (x, z)$ and dispersion equation,

$$\varepsilon_{xx}(\varepsilon_{zz} - k^2 c^2 / \omega^2) = \varepsilon_{xz}. \quad (11)$$

Of course, we would like to have $\varepsilon_{xz} = 0$, which would ease our life and give a simpler, two branches dispersion equation,

$$\begin{aligned} \varepsilon_{xx} &= 0, \\ \varepsilon_{zz} &= k^2 c^2 / \omega^2. \end{aligned} \quad (12)$$

Eq. (12) has been frequently used in the literature to study the Filamentation instability⁴. It defines purely transverse waves with $\mathbf{E}_1 \in (z)$, that is, \parallel to the flow. The problem is that these papers never say they assume $\varepsilon_{xz} = 0$. In general, they are wrong.

I wrote “in general”, because on rare occasions, they study settings for which truly, $\varepsilon_{xz} = 0$. Which are they? Remember that even if we now focus on $\mathbf{k} = (k_x, 0, 0)$, this tensor element

⁴See Bret *et al.*, Phys. Plasmas, **14**, 032103 (2007).

still depends on the beam and plasma distribution functions. A detailed study⁵ shows ε_{xz} strictly vanishes *only if* our counter streaming species are *perfectly symmetric*.

So, unless our density ratio is 1, and we have the same temperatures on the beam and the plasma, the same Lorentz factors, the same... everything, the correct dispersion equation is Eq. (11), *not* (12).

Cold Analysis - Relativistic effects

What we've said is so far non-relativistic. Still in the fluid model, the main relativistic effect is displayed when linearizing the Euler equation. The relativistic Euler equation reads,

$$\frac{\partial \mathbf{p}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{p} = q \left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right), \quad \mathbf{p} = \gamma m \mathbf{v}. \quad (13)$$

Its two linearized versions are,

$$\begin{aligned} im(\mathbf{k} \cdot \mathbf{v}_0 - \omega) \mathbf{v}_1 &= q \left(\mathbf{E}_1 + \frac{\mathbf{v}_0 \times \mathbf{B}_1}{c} \right), \quad \text{non-relativistic,} \\ im(\mathbf{k} \cdot \mathbf{v}_0 - \omega) \left(\gamma_0 \mathbf{v}_1 + \gamma_0^3 \frac{\mathbf{v}_1 \cdot \mathbf{v}_0}{c^2} \mathbf{v}_0 \right) &= q \left(\mathbf{E}_1 + \frac{\mathbf{v}_0 \times \mathbf{B}_1}{c} \right), \quad \text{relativistic.} \end{aligned} \quad (14)$$

Everything is in the anisotropic linearization of $\gamma_0 \mathbf{v}$ around \mathbf{v}_0 . We see above that for a small motion along the flow, the relativistic mass increase goes like γ_0^3 . But for small motion normal to the flow, $\mathbf{v}_1 \cdot \mathbf{v}_0 = 0$ and the mass increase only goes with γ . This of course, adds a level of complexity to the general calculation, as Eq. (10) from *Plasma Talk 8* for \mathbf{v}_1 is even more involved.

For the filamentation instability, we have $\mathbf{v}_1 \cdot \mathbf{v}_0 = 0$, and we find we can just formally replace $m \rightarrow \gamma m$. Assuming a cold beam with density n_b , Lorentz factor γ_b , and cold plasma electrons with density n_p and Lorentz factor γ_p , the tensor elements are⁶,

$$\begin{aligned} \varepsilon_{xx} &= 1 - \frac{\alpha}{x^2 \gamma_b} - \frac{1}{x^2 \gamma_p}, \\ \varepsilon_{yy} &= 1 - \frac{\alpha}{x^2 \gamma_b} - \frac{1}{x^2 \gamma_p}, \\ \varepsilon_{zz} &= 1 - \frac{\alpha}{x^2 \gamma_b} - \frac{\alpha Z^2}{x^4 \gamma_b} - \frac{1}{x^2 \gamma_p} - \frac{\alpha^2 Z^2}{x^4 \gamma_p}, \\ \varepsilon_{xz} &= \frac{\alpha Z}{x^3 \gamma_p} \left(\frac{1}{\gamma_p} - \frac{1}{\gamma_b} \right), \end{aligned} \quad (15)$$

with again,

$$x = \frac{\omega}{\omega_{pp}}, \quad Z = \frac{kv_b}{\omega_{pp}}, \quad \alpha = \frac{n_b}{n_p}. \quad (16)$$

⁵*Ibid.*

⁶*Ibid.*

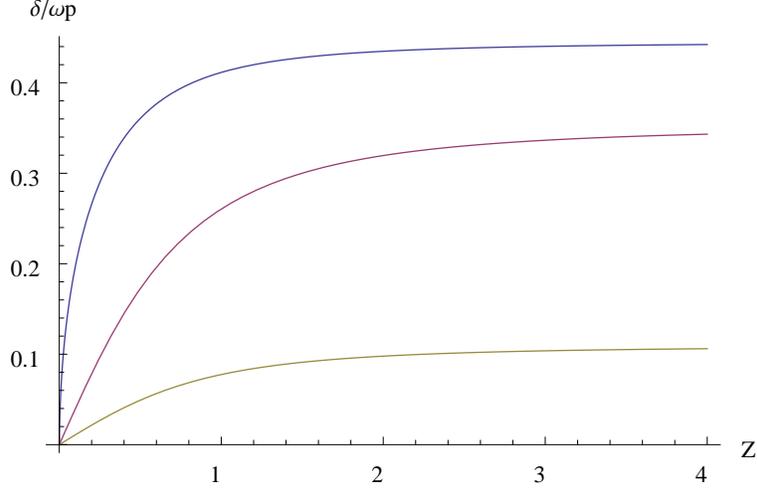


Figure 12: Filamentation instability growth-rate for density ratios $\alpha = 1, 0.5$ and 0.1 , from higher to lower curves respectively. The beam Lorentz factor is $\gamma_b = 10$.

The numerical resolution of Eq. (11), when plugging the tensor elements above, yields the growth-rate curves pictured on Fig. 12. As evidenced, the growth-rate just saturates for large Z . A trick to recover the large Z growth-rate, consists in extracting the coefficient a_n of Z^n in the polynomial dispersion equation, as $a_n = 0$ is the asymptotic dispersion equation for $Z \rightarrow \infty$. Doing so, one finds a zero real frequency and

$$\lim_{Z \rightarrow \infty} \frac{\delta}{\omega_{pp}} = \frac{v_b}{c} \sqrt{\frac{\alpha}{\gamma_b}}, \quad \alpha \ll 1, \quad (17)$$

$$= \frac{v_b}{c} \sqrt{\frac{2}{\gamma_b}}, \quad \alpha = 1, \quad (18)$$

where the agreement with Eq. (6) of *Plasma Talk 8* can be checked. Note that for $\alpha = 1$, the tensor elements (15) simplify substantially. Equation (12) for unstable modes is valid and reads,

$$x^2 - \frac{2}{\gamma_b^3} - \frac{2Z_x^2}{x^2\gamma_b} = \frac{Z_x^2}{\beta^2}, \quad (19)$$

which can be solved exactly.

We can follow the same line of reasoning for the “wrong” transverse dispersion equation (12), in order to check its inaccuracy. The exact result is for any α ,

$$\lim_{Z \rightarrow \infty} \frac{\delta_T}{\omega_{pp}} = \beta \sqrt{\frac{\alpha(\alpha\gamma_b + \gamma_p)}{\gamma_b\gamma_p}} = \beta \sqrt{\frac{\alpha(\alpha\gamma_b + 1)}{\gamma_b}}, \quad \alpha \ll 1, \quad (20)$$

$$= \beta \sqrt{\frac{2}{\gamma_b}}, \quad \alpha = 1. \quad (21)$$

As expected, the result for the symmetric case $\alpha = 1$ is the same. But for the diluted beam regime $\alpha \ll 1$, the “transverse” growth-rate δ_T differs from the exact one by,

$$\delta_T = \delta \sqrt{1 + \alpha \gamma_b}, \quad (22)$$

so that the transverse calculation overestimates the growth-rate by a factor which can be arbitrarily large.

10 Oblique Modes

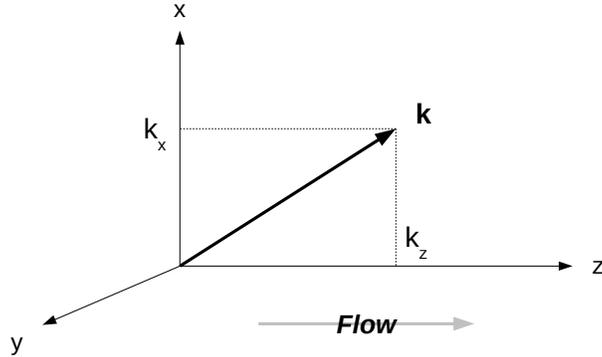


Figure 13: Axis conventions and setup.

The electrostatic approximation

We now come to these fast growing “oblique” modes in the relativistic regime. They are found for both $k_{\parallel} \neq 0$ and $k_{\perp} \neq 0$, thus their name.

Let’s remind the general dispersion equation for a set-up like to one pictured on Fig. 13. From Eqs. (1,2) of *Plasma Talk 9*, we have the dispersion equation¹

$$\det \mathbf{T}(\mathbf{k}, \omega) = 0, \quad (1)$$

with,

$$\mathbf{T} = \begin{vmatrix} \eta^2 \varepsilon_{xx} - k_z^2 & 0 & \eta^2 \varepsilon_{xz} + k_z k_x \\ 0 & \eta^2 \varepsilon_{yy} - k^2 & 0 \\ \eta^2 \varepsilon_{xz} + k_x k_z & 0 & \eta^2 \varepsilon_{zz} - k_x^2 \end{vmatrix}, \quad (2)$$

where $\eta = \omega/c$ and $\varepsilon_{\alpha\beta}$ is given by Eq. (8) of the Review Paper. We could just go on with this expression, plug some distribution functions for the beam and the plasma, and solve the dispersion equation. Doing so, we would realize something important for these oblique modes: unless we’re really close the $k_{\parallel} = 0$, these modes have $\mathbf{k} \times \mathbf{E} \sim \mathbf{0}$. That had already been noticed long ago in the first papers on the topic², for the cold case. That’s been confirmed recently for the hot case with various distribution functions³, but to my knowledge, it hasn’t been proved from the formalism.

It is then fruitful to assume $\mathbf{k} \times \mathbf{E} = \mathbf{0}$. Although this approximation breaks down for \mathbf{k} near the normal direction, it has so far been found valid for the fastest growing oblique mode. The approximation is called the “electrostatic” or “longitudinal” approximation.

¹There are some general symmetry requirements on the distribution function. See Review Paper.

²Bludman *et al.*, *Phys. Fluids* **3**, 741 & 747 (1960). Fainberg *et al.*, *Sov. Phys. JETP* **30**, 528 (1970).

³Bret *et al.*, *Phys. Rev. E* **70**, 046401 (2004) & *Phys. Rev. E* **81**, 036402 (2010).

Poisson's equation can still deliver a dispersion equation, but in a slighter intricate way because of relativistic effects (a derivation from Ampère's equation, similar to the filamentation one, is exposed in the Appendix). We simply go through the calculations of *Plasma Talk* 8 & 9, assuming at each steps $\mathbf{k} \times \mathbf{E}_1 = \mathbf{0}$, implying $\mathbf{B}_1 = \mathbf{0}$ as well.

The relativistic linearized conservation and Euler equations give for each species,

$$\begin{aligned} n_1 &= n_0 \frac{\mathbf{k} \cdot \mathbf{v}_1}{\omega - \mathbf{k} \cdot \mathbf{v}_0} \\ m \left(\gamma_0 \mathbf{v}_1 + \gamma_0^3 \frac{\mathbf{v}_1 \cdot \mathbf{v}_0}{c^2} \mathbf{v}_0 \right) &= q \left(\mathbf{E}_1 + \frac{\mathbf{v}_0 \times \mathbf{B}_1}{c} \right). \end{aligned} \quad (3)$$

Solving these two equations gives for the perturbed density,

$$n_1 = i \frac{k_z E_{1z} + \gamma^2 k_x E_{1x}}{(\omega - \mathbf{k} \cdot \mathbf{v}_0)^2}. \quad (4)$$

Inputs from each species are then merged through Poisson's equation,

$$\mathbf{k} \cdot \mathbf{E}_1 = \omega_{pb}^2 \frac{k_z E_{1z} + \gamma_b^2 k_x E_{1x}}{(\omega - \mathbf{k} \cdot \mathbf{v}_{0b})^2} + \omega_{pp}^2 \frac{k_z E_{1z} + \gamma_p^2 k_x E_{1x}}{(\omega - \mathbf{k} \cdot \mathbf{v}_{0p})^2}, \quad (5)$$

which may be put under the form $\mathbf{W} \cdot \mathbf{E}_1 = 0$, where the *vector* \mathbf{W} reads,

$$\mathbf{W} = \begin{pmatrix} k_x \\ 0 \\ k_z \end{pmatrix} - \frac{\omega_{pb}^2}{(\omega - \mathbf{k} \cdot \mathbf{v}_{0b})^2} \begin{pmatrix} \gamma_b^2 k_x \\ 0 \\ k_z \end{pmatrix} - \frac{\omega_{pp}^2}{(\omega - \mathbf{k} \cdot \mathbf{v}_{0p})^2} \begin{pmatrix} \gamma_p^2 k_x \\ 0 \\ k_z \end{pmatrix}. \quad (6)$$

Now, $\mathbf{k} \parallel \mathbf{E}_1$ and $\mathbf{W} \cdot \mathbf{E}_1 = 0$, implies $\mathbf{W} \cdot \mathbf{k} = 0$, which gives,

$$1 - \frac{k_z^2 + k_x^2 \gamma_b^2}{k_z^2 + k_x^2} \frac{\omega_{pb}^2 / \gamma_b^3}{(\omega - k_z v_b)^2} - \frac{k_z^2 + k_x^2 \gamma_p^2}{k_z^2 + k_x^2} \frac{\omega_{pp}^2 / \gamma_p^3}{(\omega - k_z v_p)^2} = 0. \quad (7)$$

Note that in this longitudinal approximation, there are *no* oblique effects for $\gamma_b = \gamma_p = 1$. A generalization of the result to the kinetic level is not as obvious as in the 1D theory for the two-stream instability, precisely because we are not 1D. The kinetic equation reads⁴,

$$0 = 1 + \frac{4\pi q^2}{k^2} \int \frac{\mathbf{k} \cdot \partial f_0(\mathbf{p}) / \partial \mathbf{p}}{\omega - \mathbf{k} \cdot \mathbf{v}} d^3 p, \quad \mathbf{p} \frac{m \mathbf{v}}{\sqrt{1 - v^2/c^2}}. \quad (8)$$

It should not be very difficult to derive intuitively Eq. (8) from Eq. (7) summing the beamlets contributions, as we did for the 1D case. In terms of the usual dimensionless variables,

$$x = \frac{\omega}{\omega_{pp}}, \quad Z = \frac{k v_b}{\omega_{pp}}, \quad \alpha = \frac{n_b}{n_p}, \quad (9)$$

and using $v_p = -\alpha v_b$, Eq. (7) reads,

$$0 = 1 - \underbrace{\frac{Z_z^2 + \gamma_b^2 Z_x^2}{Z_z^2 + Z_x^2} \frac{\alpha / \gamma_b^3}{(x - Z_z)^2}}_{beam} - \underbrace{\frac{Z_z^2 + \gamma_p^2 Z_x^2}{Z_z^2 + Z_x^2} \frac{1 / \gamma_p^3}{(x + Z_z \alpha)^2}}_{plasma}. \quad (10)$$

⁴The quantity equal to 0 here is called the dispersion "function". See S. Ichimaru, *Basic Principles of Plasma Physics: A Statistical Approach*, Chapter 3.

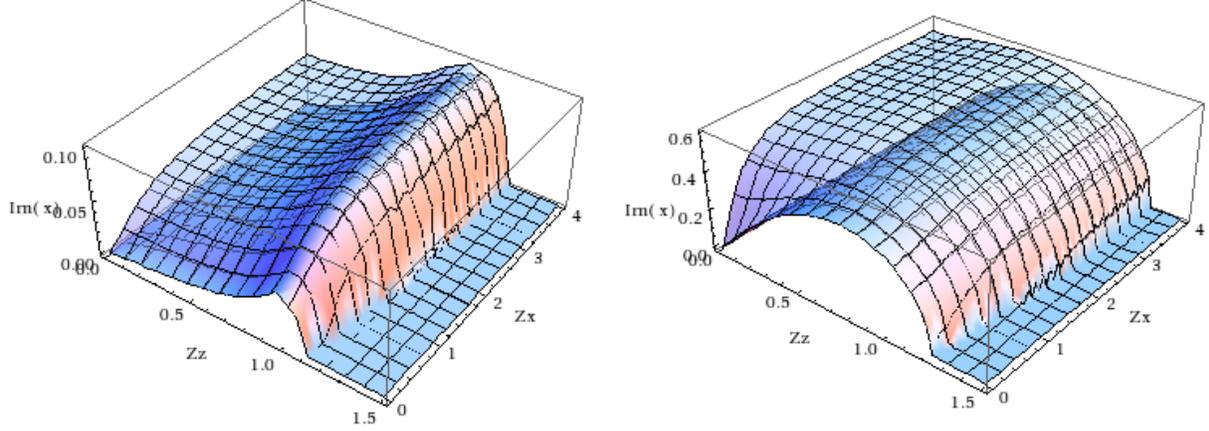


Figure 14: LEFT: Exact growth rate from dispersion equation (18) (transparent) vs. longitudinal for $\alpha = 10^{-2}$ and $\gamma_b = 3$. RIGHT: Same for $\alpha = 1$ and $\gamma_b = 1.1$.

Diluted beam

For $\alpha \ll 1$, $\gamma_p \sim 1$ and Eq. (10) reads,

$$0 = 1 - \frac{Z_z^2 + \gamma_b^2 Z_x^2}{Z_z^2 + Z_x^2} \frac{\alpha/\gamma_b^3}{(x - Z_z)^2} - \frac{1}{(x + Z_z \alpha)^2}. \quad (11)$$

This equation is very similar to the one we found for the diluted two-stream case (non-relativistic). We formally deal with a diluted beam of equivalent density ratio,

$$\alpha' = \frac{\alpha}{\gamma_b^3} \frac{Z_z^2 + \gamma_b^2 Z_x^2}{Z_z^2 + Z_x^2}. \quad (12)$$

The maximum growth rate will be found for $Z_z \sim 1$, and the frequency of the unstable mode reads,

$$\begin{aligned} \Im(x) &= \frac{\sqrt{3}}{2^{4/3}} \frac{\alpha^{1/3}}{\gamma_b} \left(\frac{1 + \gamma_b^2 Z_x^2}{1 + Z_x^2} \right)^{1/3}, \\ \Re(x) &= 1 - \frac{1}{2^{4/3}} \frac{\alpha^{1/3}}{\gamma_b} \left(\frac{1 + \gamma_b^2 Z_x^2}{1 + Z_x^2} \right)^{1/3}. \end{aligned} \quad (13)$$

The growth rate (13) displays THE oblique effect: For perp components of the wave vector such that $\gamma_b^2 Z_x^2 \gg 1$, we switch from a γ_b^{-1} to a $\gamma_b^{-1/3}$ scaling.

The validity of the longitudinal approximation is tested on Fig. 14 where it is compared to the exact solution. As expected, it breaks down for small Z_z while the exact calculation renders the filamentation growth rate as well.

Fixing $Z_z = 1$, we can compare the exact solution, the longitudinal result and Eq. (13) along the perp Z_x direction. The result is displayed on Fig. 15.

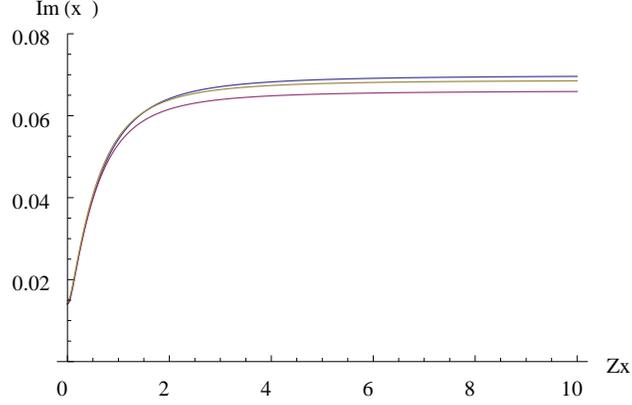


Figure 15: Comparison of the exact growth rate (top, blue curve), the longitudinal result (lower, purple curve) and Eq. (13) [middle, yellow curve] along the perp Z_x direction at fixed $Z_z = 1$. The density ratio is $\alpha = 0.01$ and the beam Lorentz factor $\gamma_b = 10$.

Symmetric beams

For $\alpha = 1$, $\gamma_b = \gamma_p \equiv \gamma$ and Eq. (10) reads,

$$0 = 1 - \frac{1}{(x - Z_z)^2 \Gamma^3} - \frac{1}{(x + Z_z)^2 \Gamma^3}, \quad \Gamma = \gamma \left(\frac{Z_z^2 + Z_x^2}{Z_z^2 + \gamma^2 Z_x^2} \right)^{1/3}. \quad (14)$$

With an equation formally equivalent to the one studied for the two-stream symmetric case in *Plasma Talk 7*. Although the equation above can be exactly solved for x , studying the fastest growing Z_z for any given Z_x is difficult because both are eventually inside Γ . Once the equation is solved, we can however look at the large Z_x limit of the growth rate which reads,

$$\delta_{Z_x \rightarrow \infty}^2 = \frac{1 + Z_z^2 \gamma_b - \sqrt{1 + 4Z_z^2 \gamma_b}}{\gamma_b}, \quad (15)$$

reaching the extremum,

$$\delta_{m, Z_x \rightarrow \infty} = \frac{1}{2\sqrt{\gamma_b}}, \quad \text{for} \quad Z_{z,m} = \frac{\sqrt{3}}{2\sqrt{\gamma_b}}. \quad (16)$$

Exact dispersion equation

Without the longitudinal approximation, and for arbitrarily oriented \mathbf{k} 's, we are back to the determinant of the tensor (2) for the dispersion equation. It has two branches corresponding to the two factors of the determinant,

$$\varepsilon_{yy} = k^2 c^2 / \omega^2, \quad \mathbf{E}_1 \in (y), \quad (17)$$

$$(\eta^2 \varepsilon_{xx} - k_z^2)(\eta^2 \varepsilon_{zz} - k_x^2) = (\eta^2 \varepsilon_{xz} - k_x k_z)^2, \quad \mathbf{E}_1 \in (x, z). \quad (18)$$

The second branch therefore holds the two-stream, the oblique and the filamentation instabilities. As evidenced by the exact plot on Figs. 14, there is a continuous transition from two-stream to filamentation modes, probably linked to a common underlying physics. *Any ideas ?*

Cold hierarchy

We may finally establish the hierarchy of modes for the cold regime in the (α, γ_b) phase space. The competing modes, with their variation from $\alpha \ll 1$ to 1, are

$$\begin{aligned}
 \text{Two - stream, } \mathfrak{I}(x) &= \frac{\sqrt{3}}{2^{4/3}} \frac{\alpha^{1/3}}{\gamma_b} \rightarrow \frac{1}{2\gamma_b^{3/2}}, \\
 \text{Oblique, } \mathfrak{I}(x) &= \frac{\sqrt{3}}{2^{4/3}} \left(\frac{\alpha}{\gamma_b}\right)^{1/3} \rightarrow \frac{\sqrt{3}}{2\gamma_b^{1/2}}, \\
 \text{Filamentation, } \mathfrak{I}(x) &= \frac{v_b}{c} \sqrt{\frac{\alpha}{\gamma_b}} \rightarrow \frac{v_b}{c} \sqrt{\frac{2}{\gamma_b}}.
 \end{aligned} \tag{19}$$

The Two-stream case is quickly settled: it is always slower than the oblique unless $\gamma_b = 1$. In the cold regime, the two-stream instability *never* governs the unstable spectrum⁵.

We are thus left comparing oblique and filamentation modes. For the diluted regime, the γ_b scaling clearly favors the oblique. Situation is more involved near the symmetric regime. As evidenced on Fig. 14 RIGHT, the longitudinal approximation gives the good order of magnitude for the growth rate, but is not enough to render the “fine structure” of the problem.

What is this “fine structure”? In the diluted beam regime, we clearly have a local extremum for oblique vectors corresponding to oblique modes. When approaching the symmetric regime, this may no longer be the case. To evidence this, we’ve plotted on Fig. 16 LEFT the growth rate at $Z_x \rightarrow \infty$ for different parameters. For $\alpha = 0.3$, we clearly find an oblique extremum, which turns to be the dominant mode. But in the symmetric case $\alpha = 1$, the local extremum disappears, giving rise to a monotonous behavior and a system governed by filamentation.

As a consequence, the oblique/filamentation frontier has to be determined numerically for large density ratios. The resulting hierarchy plot can be found on Fig. 16, RIGHT. The frontier position for $\alpha = 1$ can be determined exactly from the dispersion equation at $Z_x \rightarrow \infty$. At $\alpha = 1$, the equation can be solved, and the local oblique extremum vanishes when the second derivative of the growth rate at $Z_z = 0$ vanishes. For $\alpha \neq 1$, this second derivative vanishes on the blue line. Note the frontier gets closer to $\alpha = 1$ for large γ_b ’s, with a convergence numerically found like $\gamma_b^{-0.395}$.

More detailed in the Review Paper, Section IV-A.

⁵We’ll see later that temperature effects change this. This is why the two-stream instability can be observed in some real systems.

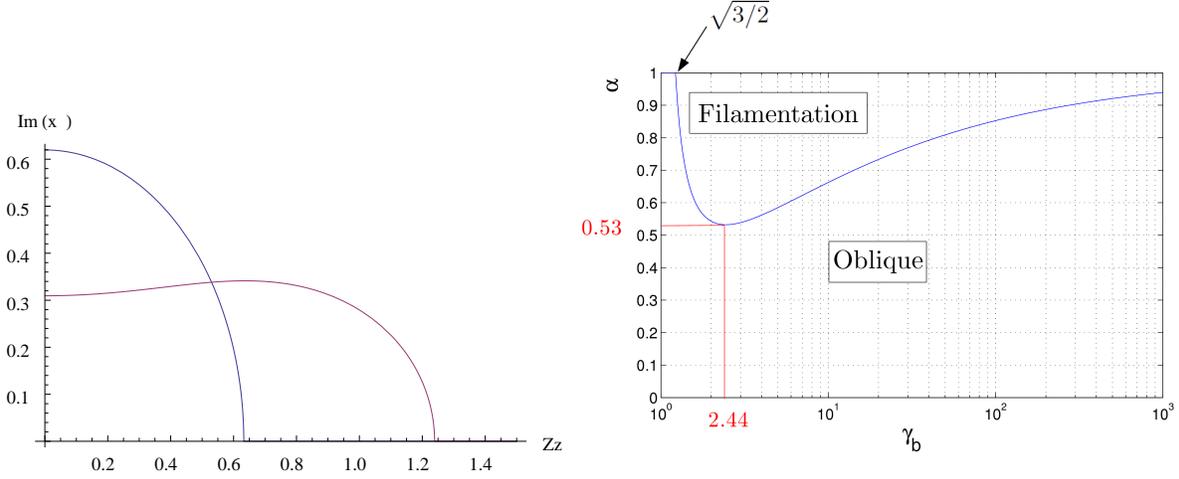


Figure 16: LEFT: Growth rate $Z_x \rightarrow \infty$ in terms of the parallel wave vector Z_z , for $\gamma_b = 5$, and $\alpha = 0.3$ and 1 (lower and upper curves respectively). The local oblique extremum vanishes when approaching the symmetric case. RIGHT: Dominant mode in terms of (γ_b, α) .

Appendix

We here derive the dispersion equation (7) from Maxwell-Ampère equation. The equation we used to merge information from each species, namely Eq. (12) from *Plasma Talk 8* simplifies in the longitudinal approximation,

$$\frac{c^2}{\omega^2} \mathbf{k} \times (\mathbf{k} \times \mathbf{E}_1) + \mathbf{E}_1 + \frac{4i\pi}{\omega} \mathbf{J}_1 = 0. \quad (20)$$

From Eqs. (3,4), the first order current is expressed in terms of \mathbf{E}_1 , leading again to a tensorial equation of the form,

$$\mathbf{T} \cdot \mathbf{E}_1 = \mathbf{0}. \quad (21)$$

If we have but two counter-streaming species (beam + plasma), the tensor reads⁶,

$$\mathbf{T} = \begin{pmatrix} \omega + \frac{\omega_{pb}^2}{\gamma_b(k_z v_b - \omega)} + \frac{\omega_{pp}^2}{\gamma_p(k_z v_p - \omega)} & 0 & 0 \\ 0 & 0 & 0 \\ -k_x \left(\frac{v_b \omega_{pb}^2}{\gamma_b(k_z v_b - \omega)^2} + \frac{v_p \omega_{pp}^2}{\gamma_p(k_z v_p - \omega)^2} \right) & 0 & \omega \left(1 - \frac{\omega_{pb}^2}{\gamma_b^3(k_z v_b - \omega)^2} - \frac{\omega_{pp}^2}{\gamma_p^3(k_z v_p - \omega)^2} \right) \end{pmatrix}. \quad (22)$$

The summing of elements from each species is here obvious again. Note that when \mathbf{k} is aligned with the main axis, that is $k_\perp = 0$ or $k_\parallel = 0$, the respective orientation of \mathbf{k} and \mathbf{E}_1 is easily determined, because \mathbf{E}_1 is also found along the very same main axis. Things are here different because the orientation of \mathbf{k} is arbitrary while the “easy” axis of our tensor are still the main ones.

⁶Interestingly, it is not symmetric. I don’t understand why. I did check you don’t find the correct result if you artificially add the missing element [1,3] to make it symmetric.

Assume we have \mathbf{E}_1 fulfilling Eq. (21) and $\mathbf{E}_1 \parallel \mathbf{k}$. Because \mathbf{T} is a linear operator, that implies $\mathbf{T} \cdot \mathbf{k}$ is also the null vector:

$$\mathbf{T} \cdot \mathbf{k} = \mathbf{0}. \quad (23)$$

The scalar product $\mathbf{k} \cdot (\mathbf{T} \cdot \mathbf{k})$ must therefore also vanish,

$$\mathbf{k} \cdot (\mathbf{T} \cdot \mathbf{k}) = 0. \quad (24)$$

The advantage is that the left-hand-side of Eq. (24) is now a *scalar*, giving us the dispersion equation for longitudinal waves with arbitrarily oriented \mathbf{k} 's. That quantity can be calculated from (22), and gives the dispersion equation (7).

11 Temperature Effects

I will quickly go through the main temperature, i.e. energy spread effects, on our instabilities. Let's first start finding out about the limits of the cold regime.

When are we no longer “cold”?

The instability process is a matter of wave-particle interaction¹. Assume a mode \mathbf{k} is exchanging energy with a group of particles. If during one growth period, all particles remain in phase with the wave, the interaction is virtually cold. The wave grows as if there was no thermal spread at all. Writing that after one growth period, the velocity spread along \mathbf{k} produces a spatial spread *smaller* than the wavelength, we find the condition for the validity of the cold model²,

$$\Delta v_{\mathbf{k}} \delta^{-1} \ll k^{-1}, \quad (1)$$

where δ is the growth rate. Note worthily, the condition is *not* homogenous throughout the \mathbf{k} space. The spread $\Delta v_{\mathbf{k}}$ and the growth rate both depend on \mathbf{k} . A given system may be virtually cold for the two-stream instability, and hot for the filamentation.

The same physical picture allows to understand the main effect of temperature. Thermal spread reduces the growth rates, precisely because if condition (1) is *not* fulfilled, the wave can exchange energy only with a fraction of the particles involved in the cold regime.

See Section III.C of the Review.

Two-stream modes

Assume a 1D beam/plasma system with a velocity distribution such as the one pictured on Fig. 17 LEFT, and define the temperature parameter,

$$\rho = \frac{V_t}{V}. \quad (2)$$

For $\rho = 0$, we have two counter-streaming symmetric beams. But it is obvious that if $\rho = 1$, the two distributions make contact, and we end up with a total distribution equivalent to an homogenous *stable* plasma at rest, with velocity spread equal to $\pm 2V$.

Indeed, the dispersion equation is easily computed, and reads in terms of the usual dimensionless parameters,

$$1 - \frac{1}{(x+Z)^2 - (\rho Z)^2} - \frac{1}{(x-Z)^2 - (\rho Z)^2} = 0. \quad (3)$$

A plot of the growth rate is pictured on Fig. 17 RIGHT, evidencing the progressive stabilization of the system for ρ approaching unity.

The same pattern holds for more realistic distribution functions. The so-called “Penrose Criterion”³ states that distribution functions are unstable if they have more than one local

¹Though there could be some issues here. See the end of the two-stream section.

²Fainberg *et al.*, *Sov. Phys. JETP* **30**, 528 (1970).

³Oliver Penrose, brother of Roger Penrose, *Phys. Fluids* **3**, 258 (1960).

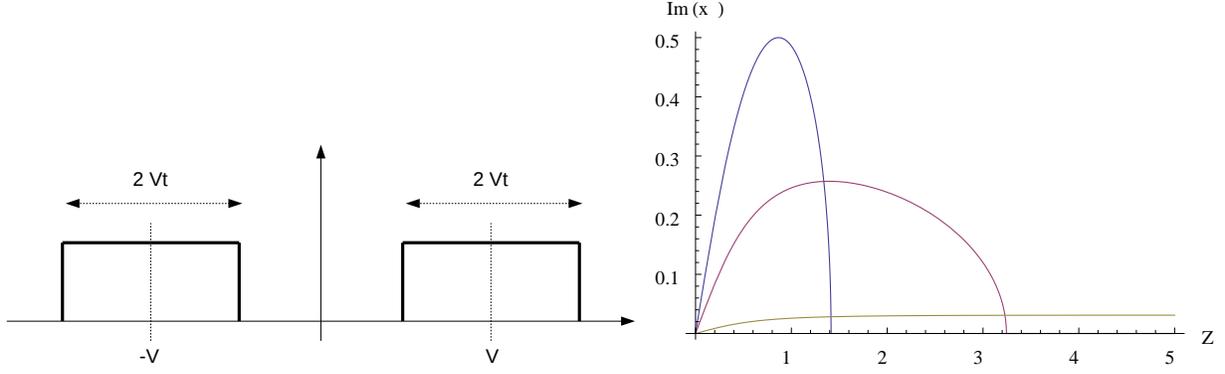


Figure 17: LEFT: Simple toy model for the stabilization of the 1D two-stream instability. RIGHT: Growth rate in terms of $Z = kV/\omega_p$ for $\rho = 0, 0.9$ and 0.999 from higher to lower curves respectively. The system is stable for $\rho > 1$.

extremum. Bottom line: for hot enough beam and/or plasma, two-stream can be stabilized, relativistic or not⁴.

Something interesting: It is tempting to relate the former criterion to the formula for Landau Damping giving a growth rate $\propto f'_0(\omega/k)$. Nevertheless, we find here unstable waves with a distribution function which derivative is almost always zero⁵! In addition, when the system is unstable for $\rho < 1$, the real part of the unstable modes is found at $\omega = 0$, so that $f'_0(\omega/k) = 0$ in our case, while there are *no* particles at $v = 0$. This is *not* an artifact of our distribution functions, because $\omega = 0$ also with two counter-streaming symmetric Maxwellian species.

I have never seen this kind of issues discussed, except in one single paper⁶. There are things left to understand...

Filamentation modes

Let's extend our toy model consisting in distributions flat up to a certain velocity ("waterbag"). Consider the 3 distribution functions pictured on Fig. 18. The shaded areas are uniformly filled with particles in velocity space.

- **A** is a counter-streaming system. Unstable to both two-stream and filamentation instabilities.
- In **B**, we just extend the parallel spread until the distributions come in contact. According to the previous paragraph, the result is two-stream stable. But the result is also anisotropic. Weibel found⁷ it is unstable to perturbations with \mathbf{k} normal to the

⁴Buschauer, *MNRAS*, **137** 99 (1977).

⁵To be more accurate, the derivative of the distribution function pictured in Fig. 17 LEFT, goes like a Dirac' δ for $v = \pm V \pm V_t$.

⁶Phys. Rev. B **43**, 14009 (1991), where the problem is pointed out, but not solved.

⁷Weibel, *Phys. Rev. Lett.*, **2**, 83 (1959).

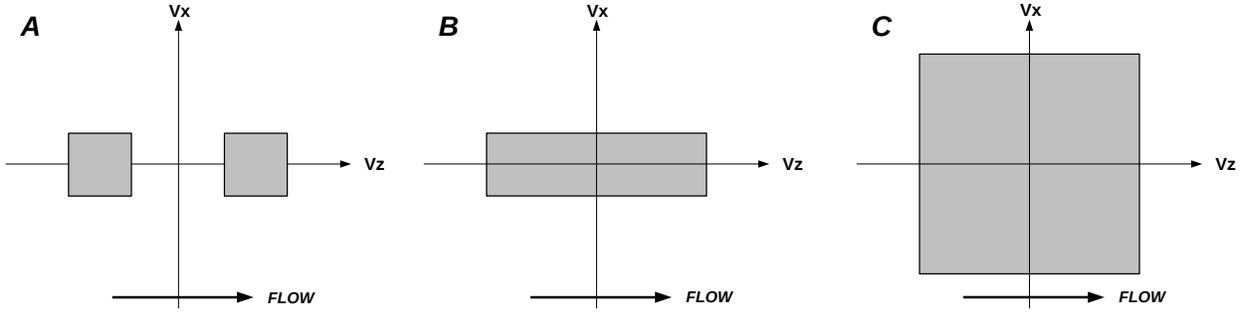


Figure 18: Distribution functions with different stability properties.

highest thermal spread: that's filamentation here. **B** is therefore two-stream stable *and* filamentation unstable.

- **C** is built from **B**, equating the spread in every directions. The result is stable.

We thus find we can “play” with temperature parameters in order to stabilize some parts of the spectrum, or all of it. The bottom line here is that filamentation can be stabilized.

Note that in Fig. 18, we carefully tailor the spreads of every species to cancel the instability. What if we just play with the beam, that is, the component around $v_z > 0$ on Fig. 18-A? The end result depends on the distribution:

- For waterbag distributions, filamentation is stabilized beyond a certain amount of *beam transverse spread*. That can be understood as follows⁸: Assume a current filament of radius $1/k$ and density n_b . The current is $I \sim qn_bk^{-2}v_b$. It creates at the surface of the filament a field $B = 2I/c k^{-1}$. A charge at the surface is therefore pulled *in* by the Lorentz force

$$F_B = q \frac{v_b B}{c} = \left(\frac{v_b}{c} \right)^2 q^2 n_b k^{-1}. \quad (4)$$

If there is no temperature, nothing prevents the filament from further pinching, which is why the instability extends up to $k = \infty$ in the cold regime. But if we're hot, kinetic pressure opposes the pinching: A little piece of filament near the surface, with volume dV and surface dS , is pulled in by $F_B n_b dV$, and pushed out by $n_b k_B T dS$. Pinching is prevented if,

$$n_b k_B T dS > F_B n_b dV \Rightarrow k > q \frac{v_b}{c} \sqrt{\frac{n_b}{k_B T}} \equiv k_{m\perp} \propto \frac{\sqrt{n_b}}{v_{tb}}, \quad (5)$$

which is the scaling found from the theory with waterbag distributions⁹ (we have used $dV \sim k^{-1} dS$, and v_{tb} is the thermal beam spread.).

⁸Silva *et al.*, *Physics of Plasmas*, **9**, 2458 (2002).

⁹Bret *et al.*, *Physical Review E*, **72**, 016403 (2005).

- For relativistic Maxwellians, it has been proved that filamentation never vanishes completely¹⁰. If T_b is the beam temperature, the maximum filamentation growth-rate scales like $T_b^{-3/2}$. In addition, this result still holds for *any* plasma temperature.

Oblique modes - general thermal “rules”

Two-stream modes are unstable up to a finite $k_{m\parallel}$, and filamentation up to yet another finite $k_{m\perp}$. How do we close the unstable domain? Two different answers according to the distribution. With waterbag, we close from, and to infinity. See Fig. 10a of the Review. With a Maxwellian, these large k oblique modes are stabilized, and we close “normally”, as pictured for example on Fig. 14 of the Review.

Oblique modes’ temperature sensitivity is intermediate between two-stream and filamentation. As evidenced in *Plasma Talk 10*, they tend to be interesting only in the relativistic regime, due to the γ_b scaling which favors them. Two thermal “rules” are useful to grasp the influence of the *beam* spread over the full spectrum:

- Parallel spread *hardly matters*. Why? Because in the relativistic regime, it takes a huge parallel *energy* spread to get a reasonable parallel *velocity* spread. All velocities are squeezed against c . See Fig. 13 of the Review.
- Two-stream does not care about the transverse spread, because particles differing only by their *transverse* velocity will equally stay tuned with a plane wave at $\mathbf{k} \parallel \textit{flow}$. For the same reason, filamentation do care.

As a result, modes are all the more affected by beam temperature than they are oblique.

We could rank, from best to worst, the 3 kinds of modes in terms of the way they “resist” the various effects:

Relativistic: Oblique \rightarrow Filamentation \rightarrow two-stream.

Density ratio: Filamentation \rightarrow Oblique/two-stream.

Beam temperature: Two-stream \rightarrow Oblique \rightarrow Filamentation.

All this ends up with the *mode hierarchy* pictured on Fig. 20 of the Review Paper.

The phase velocity diagram - Fig. 17 of the Review

A great tool to understand the physics. Plot, on the *very same graph*, the distribution functions and the phase velocities of the unstable modes. One can straightforwardly check which species are in resonance with which kind of modes.

¹⁰Gremillet, *Unpublished*, Bret *et al.*, *Physical Review E*, **81**, 036402 (2010),

12 Non-Linear Regime

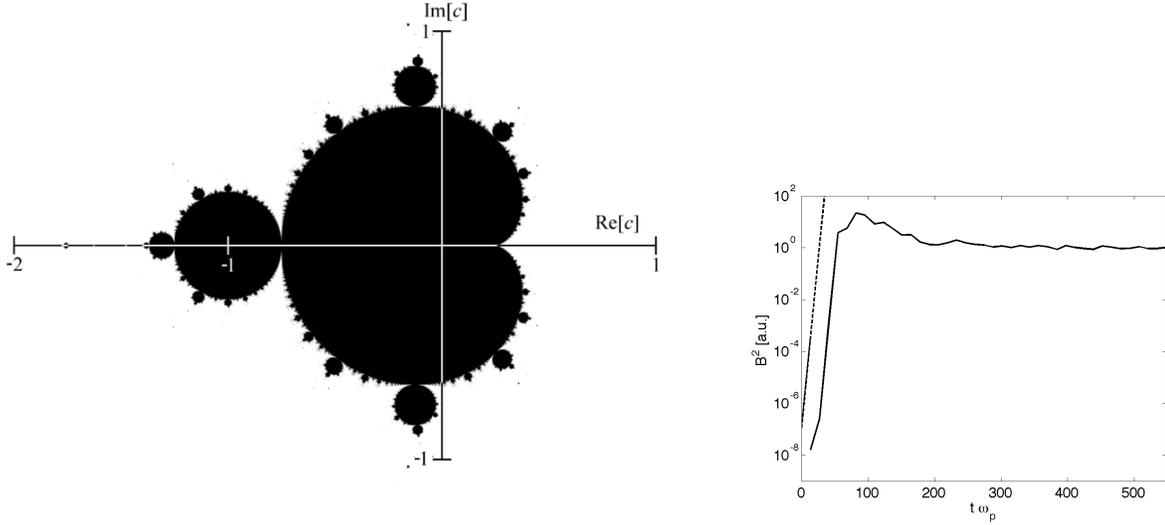


Figure 19: *Left*: The “Mandelbrot set”, defined by Eqs. (1,2). *Right*: Typical field growth extracted from a PIC simulation. Growth roughly stops when the linear exponential growth stops. The line shows the expected growth from theory.

Non-linear problems may be complicated with just 1 or 2 degrees of freedom. Think about this 2D example: take the very simple sequence¹

$$\begin{aligned} z_{n+1} &= z_n^2 + c, \\ z_0 &= c, \quad c \in \mathbb{C}, \end{aligned} \quad (1)$$

and just define the set

$$M = \left\{ c \in \mathbb{C} / \lim_{n \rightarrow \infty} |z_n| \neq \infty \right\}. \quad (2)$$

You have the famous, and incredibly complicated, fractal and everything, “Mandelbrot set” pictured on Fig. 19. People were stunned when they realized something as trivial as Eq. (1) could generate such amount of complexity².

A plasma has ∞ number of degrees of freedom. Yet, to my knowledge, something as beautiful as Fig. 19 is still lacking in plasma physics. Maybe because it’s too complicated. . .

At any rate, there’s no hope of analytically finding out about the long term evolution of our beam plasma systems in the general case. Remember it took a Fields Medal to prove non-linear Landau damping. Even for the cold case, things are not easy.

I’ll go through some results on the saturation of the various instabilities, always assuming the fastest growing mode is the *only one* excited, and that everything is *cold* at $t = 0$. The

¹Non-linear in the sense that if two sequences fulfill $z_{n+1} = z_n^2 + c$, their linear combinations do not.

²A Math-guy friend of mine once told me people would laugh at Mandelbrot, as “the guy who works on polynomial of degree 2”!

former is quite reasonable, as the most unstable mode grows *exponentially* faster than the rest. Relativistic effects help as they drive a sharper unstable spectrum, where growth rates vary rapidly from one mode to another. The later is a limitation.

The idea is to find out *when* the linear theory should break down, and to claim that growth *stops* at that point. Granted, *exponential* growth should stop there. But other kind of growth could keep on. Yet the observed field growth in PIC simulations is always like to one pictured on Fig. 19. Why?

See Section V of the *Review Paper*, and references therein.

Two-stream and oblique instabilities

Two-stream, non-relativistic

We assume a *diluted beam*. There, two-stream is resonant with the beam. Say the wave $Ee^{ikx-i\omega t}$ is growing, traveling along with the beam electrons. Electrons will start oscillating in the field at the “bouncing” frequency³,

$$\omega_b^2 = \frac{qEk}{m}. \quad (3)$$

The linear assumption that all electrons have $v = v_b$ during one growth period δ^{-1} breaks down when,

$$\omega_b = \delta \quad \Rightarrow \quad E_s = \frac{\delta^2}{\omega_p} \frac{mv_b}{q}, \quad (4)$$

giving the value of the field at saturation (I’ve set here $k \sim \omega_p/v_b$). A great by-product of E_s is the beam energy loss ΔW_b . Since the field energy can only come from the beam energy, we can write,

$$\begin{aligned} \frac{\Delta W_b}{\frac{1}{2}n_b m v_b^2} &= \frac{E_s^2/8\pi}{\frac{1}{2}n_b m v_b^2} \\ &= \frac{n_p}{n_b} \left(\frac{\delta}{\omega_p} \right)^4 \sim \left(\frac{n_b}{n_p} \right)^{1/3}, \end{aligned} \quad (5)$$

where I just replaced the growth-rate by its cold value $\frac{\sqrt{3}}{2^{4/3}}(n_b/n_p)^{1/3}\omega_p$. We could even write the energy lost is shared between the plasma and the field⁴. In such case, $\Delta W_b/W_b$ is half the result (5).

As the field grows and traps the beam electrons, they start oscillating in the wave potential. PIC people love to plot density graphs in the (x, v_x) phase-space (say x is our dimension) such as the one pictured on Fig. 20. At $t = 0$, the beam and the plasma are just two lines. In the instability phase, the plasma does not move a lot, but the beam particles start oscillating in the wave, creating “holes” in the phase-space.

³See *Plasma Talk 5* on Landau damping.

⁴Lorenzo Sironi told me you see this in the PICs. But why?

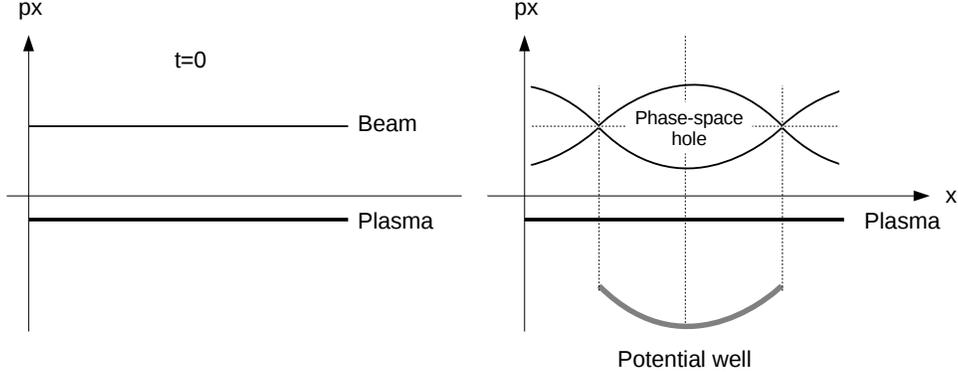


Figure 20: Phase-space hole.

Two-stream, Relativistic

A naive reasoning gives the right answer. As the oscillatory trapping motion is along the flow, we replace $m \rightarrow \gamma_b^3 m$ in the bouncing frequency (3). This gives the field at saturation⁵,

$$E_s = \frac{\delta^2}{\omega_p} \frac{\gamma_b^3 m v_b}{q}. \quad (6)$$

The relative energy loss is then computed as in Eq. (5), replacing the growth-rate by its relativistic value. It reads,

$$\frac{\Delta W_b}{n_b \gamma_b m c^2} \sim \gamma_b \beta^2 \left(\frac{n_b}{2n_p} \right)^{1/3}. \quad (7)$$

Of course, this quantity has to remain small since linear regime means unperturbed trajectories, that is, $\Delta W_b \ll W_b$. The energy loss eventually relies on the parameter S with⁶

$$S = \gamma_b \beta^2 \left(\frac{n_b}{2n_p} \right)^{1/3}. \quad (8)$$

If you do the “clean” calculation going to the wave frame, like in the footnote, you need the beam dynamic in that frame to be non-relativistic in order to compute easily a bouncing

⁵Let's denote quantities *in the wave frame* with a prime. If the beam dynamic in the wave frame is non-relativistic, we have

$$\omega_b'^2 = \frac{q E k'}{m},$$

where the field E does not need prime as it is parallel to the motion. The bouncing frequency in the lab frame is simply $\omega_b = \omega_b' / \gamma_b$. In addition, $k = \gamma_b k'$ since in the wave frame, $\omega' = 0$. We thus find

$$\omega_b^2 = \frac{q E k}{\gamma_b^3 m},$$

retrieving the $m \rightarrow \gamma_b^3 m$ “rule”.

⁶See *Review Paper*.

frequency. Doing so requires $S \ll 1$, as explained in the Review. For arbitrary S 's, one has

$$\frac{\Delta W_b}{W_b} \sim \frac{S}{(S+1)^{5/2}}, \quad (9)$$

yielding a maximum energy loss in the linear phase

$$\frac{\Delta W_b}{W_b}_{max} = \frac{6}{25} \sqrt{\frac{3}{5}} \sim 0.18 \quad \text{for} \quad S = \frac{2}{3}. \quad (10)$$

Oblique instability

Poorly known. What is known is that Eq. (9) roughly works until $S \sim 0.45$. For larger S 's, $\Delta W_b/W_b$ seems quite insensitive to S , and remains around 0.18 given by Eq. (10).

Filamentation instability

There are 3 different ways to evaluate the field at saturation!

1. **Cyclotron frequency = growth rate.** Filamentation instability grows magnetic field. Such field \mathbf{B} affects particles on a time scale given by the cyclotron frequency,

$$\omega_c = \frac{qB_1}{\gamma_b m c}. \quad (11)$$

Writing again that the linear regime keeps on until $\omega_c = \delta$, we get,

$$B_{s1} = \frac{\gamma_b m c}{q} \delta. \quad (12)$$

If one tries to compute the relative energy loss $\frac{B_{s1}^2}{8\pi} / n_b \gamma_b m c^2$, the cold result gives a factor of order β^2 , without *any* more scaling in γ_b , and almost none in $\alpha = n_b/n_p$ (see Table 1). The conclusion is that estimating the energy loss requires a finer calculation than this one, and that the result is quite stable in terms of these variables.

2. **Bouncing frequency = growth rate.** Historically, the field at saturation has rather been evaluated this way. With $\mathbf{v}_{0b} = (0, 0, v_b)$ and $\mathbf{k} = (k_x, 0, 0)$, the growing magnetic field reads $\mathbf{B}_1 = (0, B_1 \sin k_x x, 0)$. At first order, Newton's law projected on the x axis gives,

$$\gamma_b m \frac{d^2 x}{dt^2} = q B_1 \frac{v_b}{c} \sin k_x x. \quad (13)$$

Particles at $x \sim 0$ oscillate at,

$$\omega_b^2 = \frac{q B_1 v_b k_x}{\gamma_b m c}. \quad (14)$$

Note that we assumed there's only a \mathbf{B} field here. We know that unless the system is strictly symmetric, it's wrong. Here again, we can claim exponential growth keeps on while the motion is almost unperturbed, that is until

$$\frac{q B_1 v_b k_x}{\gamma_b m c} = \delta^2 \quad \Rightarrow \quad B_{s2} = \frac{\gamma_b m c}{q} \frac{\delta^2}{v_b k_x}. \quad (15)$$

Criteria	B_s	$B_s, mc\omega_p/q$ units	$\Delta W_b/W_b$
Cyclotron frequency = growth-rate	$\frac{\gamma_b mc}{q} \delta$	$\beta[\alpha(\alpha+1)\gamma_b]^{1/2}$	$\beta^2(\alpha+1)/2$
Bouncing frequency = growth-rate	$\frac{\gamma_b mc}{q} \frac{\delta^2}{v_b k_x}$	$\beta\alpha(\alpha+1)$	$\beta^2\alpha(\alpha+1)^2/2\gamma_b$
Larmor radius = characteristic k_c^{-1}	$\frac{\gamma_b mc}{q} v_0 k_c$	$\beta\gamma_b$	$\beta^2\gamma_b/2\alpha$

Table 1: Summary of the 3 ways to evaluate the field at saturation B_s for the filamentation instability, *considering* $k_\perp = \omega_p/c$. Results for the cold case. Taking $k_\perp \propto \gamma_b^{-1/2}$, the 3 criteria give the same scaling for $\alpha = 1$ [1].

Comparing Eqs. (15,12) gives,

$$B_{s2} = B_{s1} \frac{\delta}{v_b k_x}, \quad (16)$$

so that both estimates give the same result *only* with $k_x = \delta/v_b$. For example, with the symmetric cold case where $\delta = \omega_p \beta \sqrt{2/\gamma_b}$, that implies

$$k_x = \frac{\omega_p}{c} \sqrt{\frac{2}{\gamma_b}}. \quad (17)$$

For non-relativistic setting, the k_x is the typical expected one. For γ_b , very recent *cold* PIC's [1] found indeed that it *is* the fastest growing k_\perp . Why exactly, as the cold growth rate just saturates at large k_\perp 's?

3. **Larmor radius = characteristic** $1/k_\perp$. Equating the Larmor radius of an electron in a field \mathbf{B} to the characteristic k_c of the insta gives,

$$B_{s3} = \frac{\gamma_b mc}{q} v_0 k_c = B_{s1} \frac{v_0 k_c}{\delta}. \quad (18)$$

The 3 results are summarized in Table 1, considering the cold symmetric case, and taking $k = \omega_p/c$ for the typical k_\perp . Taking $k_\perp \propto \gamma_b^{-1/2}$, the 3 criteria give the same scaling for $\alpha = 1$ [1].

Fate of the filaments

Opposite filaments repel, but like filaments attract. In our 3D world, filaments turn around each other, and like filaments merge. The merging process has been modeled, and successfully simulated with PICs⁷.

More realistic settings, successive instabilities

We've been so far interested in the short term evolution of the system, that is, the end of the linear phase. What's next? I'll just comment *Fig. 40 of the Review Paper*. The initial setup was:

⁷Medvedev *et al.*, *The Astrophysical Journal* **618**, L75 (2005).

Beam: Maxwellian, $n_b = n_p/10$, $\gamma_b = 3$, $T_b = 50$ keV.

Plasma: Maxwellian, $T_p = 5$ keV.

- $0 < \omega_p t < 80$: System initially governed by *oblique* modes. \mathbf{E} field grows at $0.07\omega_p$. Heating “kills” de oblique.
- $80 < \omega_p t < 160$: System switches to a *two-stream* regime. \mathbf{E} field grows at $0.016\omega_p$. Heating “kills” two-stream.
- $200 < \omega_p t < 600$: Remaining drift feeds *filamentation*. \mathbf{B} field grows at $0.005\omega_p$.

By the end of the simulation $\omega_p t \sim 600$, the beam had lost about 30% of its energy, entirely transferred to plasma electrons. Open questions:

- Is filamentation the necessary end state of every initial setup?
- Does the drift eventually ends (in the frame of the fixed ion background)? That is, is the drift energy eventually converted at 100% into heat? Sounds reasonable. Is that sure?

Indeed, the interesting question might be *how long* does it take?

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13 Ohm's law and the Biermann battery

The original 1950 paper is Ref. [1], “Über den Ursprung der Magnetfelder auf Sternen und im interstellaren Raum”, published in *Zeitschrift Naturforschung Teil A*. Cited more than 200 times, and probably read by no one but the happy few who 1) read German and 2) could access it.

■ The MHD equations are formed from the fluid equations for electrons and ions (dropping subindices),

$$\begin{aligned} \frac{\partial n}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (n\mathbf{v}) &= 0, \\ mn \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{r}} \right) &= qn \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) - \nabla p + nm\mathbf{g}. \end{aligned} \quad (1)$$

The MHD variables are defined from $n_{e,i}(\mathbf{r}, t)$, $\mathbf{v}_{e,i}(\mathbf{r}, t)$ as¹,

$$\begin{aligned} \rho(\mathbf{r}, t) &= m_i n_i + m_e n_e, \\ \mathbf{J}(\mathbf{r}, t) &= qn_i \mathbf{v}_i - qn_e \mathbf{v}_e, \\ \mathbf{V}(\mathbf{r}, t) &= \frac{1}{\rho(\mathbf{r}, t)} (n_e m_e \mathbf{v}_e + m_i n_i \mathbf{v}_i). \end{aligned}$$

Merging the fluid Euler equations for both species gives,

$$\rho \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \frac{\partial \mathbf{V}}{\partial \mathbf{r}} \right) = \frac{\mathbf{J}}{c} \times \mathbf{B} - \nabla \overbrace{(p_i + p_e)}^P + \rho \mathbf{g}. \quad (2)$$

Ohm's law is the equation giving the current \mathbf{J} . Where does it come from? What we did in *Plasma Talk 3* was to follow the basic, “business as usual” procedure: sit in the frame of the fluid locally at \mathbf{V} . There, the electric field is \mathbf{E}' . Ohm's law gives the current in the lab frame from \mathbf{E}' , as $\mathbf{J}' = \sigma \mathbf{E}'$. Because $\mathbf{E}' = \mathbf{E} + \mathbf{V} \times \mathbf{B}/c$, we get the famous (non-relativistic)

$$\mathbf{J} = \sigma \left(\mathbf{E} + \frac{\mathbf{V}}{c} \times \mathbf{B} \right). \quad (3)$$

What about $\mathbf{J}' = \sigma \mathbf{E}'$? It arises from the microscopic picture that under the action of an electric field, particles, mostly electrons, are accelerated in the direction of the field, while collisions with the ions act like a friction force². Writing something like $m\partial_t \mathbf{v} = q\mathbf{E}' - \nu \mathbf{v}$ where ν is some collision frequency, and setting $\partial_t = 0$, indeed yields $\mathbf{v} = \frac{q}{\nu} \mathbf{E}'$ and then $\mathbf{J}' = qn \frac{q}{\nu} \mathbf{E}'$. Ideal MHD assumes $\nu = 0$, giving

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{V}}{c} \times \mathbf{B} = \mathbf{0}. \quad (4)$$

¹In *Plasma Talk 3*, the MHD velocity \mathbf{V} was defined through $(m_e + m_i)\mathbf{V} = m_e \mathbf{v}_e + m_i \mathbf{v}_i$. Most books [2, 3] present definition (2) above. It is more rigorous, as it gives a $\rho \mathbf{V}$ MHD term exactly equal to the total momentum. At any rate, the difference between the two quantities is $\propto (n_e - n_i)(\mathbf{v}_e - \mathbf{v}_i)$. This is a second order quantity in the MHD regime, where electrons are expected to closely follow the ions, so that $n_e \sim n_i$ and $\mathbf{v}_e \sim \mathbf{v}_i$.

²Solid state physics call this the “Drude model”, from Paul Drude, who came up with this idea in 1900.

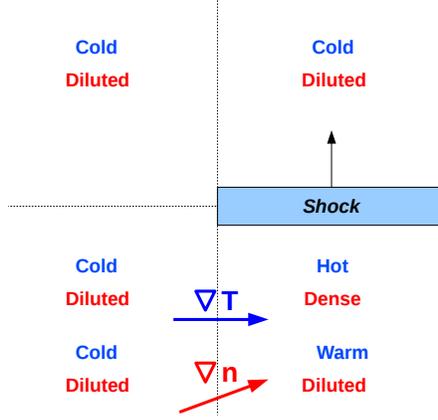


Figure 21: How a finite size shock can generate non-parallel density and temperature (hence pressure) gradients. From [4], p. 406.

■ But we inadvertently assumed many things. For example, $\partial_t = 0$ assumes \mathbf{E}' varies slowly enough with time. If variations are too fast, the stationary regime does not have enough time to set in, and some ω dependency appears. Also, we assumed particles are accelerated only by \mathbf{E}' between two collisions. What if \mathbf{B} is strong enough to curve the trajectories in between? In this case, the resistivity in the direction normal to the field is higher ($\times 1.9$) than along the field ([2] p. 28, or [4] p. 43).

Since conductivity comes from the electrons³, let's write their full Euler equation,

$$m_e n_e \left(\frac{\partial \mathbf{v}_e}{\partial t} + \mathbf{v}_e \cdot \frac{\partial \mathbf{v}_e}{\partial \mathbf{r}} \right) = q_e n_e \left(\mathbf{E} + \frac{\mathbf{v}_e}{c} \times \mathbf{B} \right) - \nabla p_e - \nu (\mathbf{v}_e - \mathbf{v}_i). \quad (5)$$

Neglecting the left-hand-side for the small electrons inertia, and setting $\nu = 0$ for ideal MHD yields,

$$\mathbf{E} + \frac{\mathbf{v}_e}{c} \times \mathbf{B} = \frac{\nabla p_e}{q_e n_e}. \quad (6)$$

According to Kulsrud [4] p. 405, the pressure term above is negligible when there is a \mathbf{B} field. **WHY?** But for small \mathbf{B} 's, or even $\mathbf{B} = 0$, you need to keep it. Inserting the electric field above in $\partial_t \mathbf{B} = -c \nabla \times \mathbf{E}$ and setting $\mathbf{v}_e \sim \mathbf{V}$ gives⁴,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + c \frac{\nabla n_e \times \nabla p_e}{q n_e^2}. \quad (7)$$

The second term is our *Biermann Battery*. There's no \mathbf{B} in there, so that it can make it from nothing. Still, the microscopic derivation shows we need ionization, just to be able to create electronic currents moved by the electronic pressure.

■ But, an equation of state usually gives $p_e(n_e)$. The gradients for p_e and n_e are thus

³The σ of $\mathbf{J}' = \sigma \mathbf{E}'$ has the mass on the denominator.

⁴We need a little drag between ions and electrons to write $\mathbf{v}_e \sim \mathbf{V}$.

likely to be parallel, so that the cross product vanishes. How do we break this? Several possibilities,

1. Try a system in rotation around z , with $\Omega(z)$. The pressure gradient reads $\nabla P \propto \Omega^2(z)\rho(r, \theta)$. It has a z component, while $\nabla\rho(r, \theta)$ has not. Those who've read the paper say Biermann considered this option.
2. Try the scheme of Figure 21. Suppose a finite size shock travels through a cold, diluted upstream. The downstream just behind the shock is dense and hot. Far downstream though, the plasma expands but remains warm, so that it is now warm *and* diluted. The figure shows well how this can generate non-parallel density and temperature (hence pressure) gradients. A *non-spherical* shock also does the job for similar reasons.

Everything is easily adapted to a system *partially* ionized. If n_n is the density of the neutral, define $\chi = n_e/(n_i + n_n)$. Eq. (7) can be adapted dividing the Battery term by $1 + \chi$ ([4], p. 406). See [7].

Kulsrud proposed this mechanism to produce Cosmic Fields from scratch in 1997 [5]. Yet, he co-authored in 1992 another paper [6] advocating spontaneous plasmas fluctuations.

Also Khanna [8], showed that a rotating BH in a plasma will always generate toroidal and poloidal magnetic fields.

Ref. [9] studied the Biermann Battery effects in Cosmological MHD Simulations of Population III Star Formation. In its own terms, “We find that the Population III stellar cores formed including this effect are both qualitatively and quantitatively similar to those from hydrodynamics-only (non-MHD) cosmological simulations”. No dynamical effects.

The Biermann battery mechanism was successfully tested in the lab in 2012 [10].

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